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TOTAL POSITIVITY FOR LOOP GROUPS II: CHEVALLEY GENERATORS

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Abstract. This is the second in a series of papers developing a theory of total positivity for loop groups. In this paper, we study infinite products of Chevalley generators. We show that the combinatorics of infinite reduced words underlies the theory, and develop the formalism of infinite sequences of braid moves, called a braid limit. We relate this to a partial order, called the limit weak order, on infinite reduced words.

The limit semigroup generated by Chevalley generators has a transfinite structure. We prove a form of unique factorization for its elements, in effect reducing their study to infinite products which have the order structure of $\mathbb{N}$. For the latter infinite products, we show that one always has a factorization which matches an infinite Coxeter element.

One of the technical tools we employ is a totally positive exchange lemma which appears to be of independent interest. This result states that the exchange lemma (in the context of Coxeter groups) is compatible with total positivity in the form of certain inequalities.

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1. Introduction

This is the second in a series of papers where we develop a theory of total positivity for the formal loop group $GL_n(\mathbb{R}((t)))$ and polynomial loop group $GL_n(\mathbb{R}[t,t^{-1}])$. We assume the reader has some familiarity with the first paper [LPI], and refer the reader to the Introduction there for the original motivation.

Let us briefly recall the main definitions from [LPI]. Suppose $A(t)$ is a matrix with entries which are real polynomials, or real power series. We associate to $A(t)$ a real

infinite periodic matrix $X = (x_{i,j})_{i,j=-\infty}^{\infty}$ satisfying $x_{i+n,j+n} = x_{i,j}$ and

$$a_{ij}(t) = \sum_{k=-\infty}^{\infty} x_{i,j+kn} t^k.$$

We call X the *unfolding* of $A(t)$. We declare that $A(t)$ is *totally nonnegative* if and only if X is totally nonnegative, that is, all minors of X are nonnegative.

Let $GL_n(\mathbb{R}((t)))$ denote the *formal loop group*, consisting of $n \times n$ matrices $A(t) = (a_{ij}(t))_{i,j=1}^n$ whose entries are formal Laurent series such that $\det(A(t)) \in \mathbb{R}((t))$ is a non-zero formal Laurent series. We let $GL_n(\mathbb{R}[t, t^{-1}]) \subset GL_n(\mathbb{R}((t)))$ denote the polynomial loop group, consisting of $n \times n$ matrices with Laurent polynomial coefficients, such that the determinant is a non-zero real number. We write $GL_n(\mathbb{R}((t)))_{\geq 0}$ for the set of totally nonnegative elements of $GL_n(\mathbb{R}((t)))$. Similarly, we define $GL_n(\mathbb{R}[t, t^{-1}])_{\geq 0}$. Let $U \subset GL_n(\mathbb{R}((t)))$ denote the subgroup of upper-triangular unipotent matrices, and let $U_{\geq 0}$ (resp. $U_{\geq 0}^{\mathrm{pol}}$) denote the semigroup of upper-triangular unipotent totally nonnegative matrices in $GL_n(\mathbb{R}((t)))$ (resp. $GL_n(\mathbb{R}[t, t^{-1}])$). In [LPI, Theorem 4.2] we explained how the analysis of $GL_n(\mathbb{R}((t)))_{\geq 0}$ and $GL_n(\mathbb{R}[t, t^{-1}])_{\geq 0}$ can to a large extent be reduced to analysis of $U_{\geq 0}$ and $U_{\geq 0}^{\mathrm{pol}}$. Proceeding with the latter, in [LPI] we showed

Theorem. *Let $X \in U_{\geq 0}$. Then X has a factorization of the form*

$$X = ZAVBW$$

where Z (resp. W) is a (possibly infinite) product of non-degenerate curls *(resp.* whirls*), A and B are (possibly infinite) products of Chevalley generators, and V is a* regular *matrix.*

The regular factor V will be studied in [LPIII].

In [LPI], we studied the factors Z and W in detail. In particular, they are uniquely determined by X, and furthermore they are infinite products of the forms $\prod_{i=1}^{\infty} N^{(i)}$ and $\prod_{i=-\infty}^{-1} M^{(i)}$, where the $N^{(i)}$ and the $M^{(i)}$ are distinguished matrices called *curls* and *whirls*. A whirl is a matrix given in infinite periodic form by $M = (m_{i,j})_{i,j=-\infty}^{\infty} = M(a_1, \ldots, a_n)$ with $m_{i,i} = 1$, $m_{i,i+1} = a_i$ and the rest of the entries equal to zero, where the indexing of the parameters is taken modulo n. Define $X^c \in U$ to be the matrix obtained by applying to $X \in U$ the transformation $x_{i,j} \mapsto (-1)^{|i-j|} x_{i,j}$, and define $X^{-c} := (X^c)^{-1}$. It was shown in [LPI, Lemma 4.5] that if $X \in U_{\geq 0}$ then $X^{-c} \in U_{\geq 0}$. A curl is a matrix N of the form $N(a_1, \ldots, a_n) := M(a_1, \ldots, a_n)^{-c}$.

In this paper we shall show (Theorem 7.11) that the factors A and B are unique, though in general they are not infinite products with the order structure of $\mathbb{N}$, but instead have a transfinite structure.

Let $e_i(a)$ denote *the affine Chevalley generators*, which differ from the identity matrix by the entry a in the i-th row and $(i+1)$-st column (in infinite periodic matrix notation). The two kinds of Chevalley generators for $n = 2$ are shown below.

$$e_1(a) = \left(\begin{array}{c|cc|cc|c} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \hline \cdots & 1 & a & 0 & 0 & \cdots \\ \cdots & 0 & 1 & 0 & 0 & \cdots \\ \hline \cdots & 0 & 0 & 1 & a & \cdots \\ \cdots & 0 & 0 & 0 & 1 & \cdots \\ \hline & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right), \quad e_2(b) = \left(\begin{array}{c|cc|cc|c} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \hline \cdots & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & b & 0 & \cdots \\ \hline \cdots & 0 & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & \cdots \\ \hline & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

Our approach is based upon the study of the map

$$e_{\mathbf{i}} : (a_1, a_2, \cdots) \longmapsto e_{i_1}(a_1)e_{i_2}(a_2)\cdots \tag{1}$$

which converges for $a_k > 0$ satisfying $\sum_k a_k < \infty$.

1.1. **Cell decomposition in the finite case.** Let $U_{\geq 0}^{\mathrm{fin}} \subset GL_n(\mathbb{R})$ denote the semigroup of totally nonnegative upper-triangular unipotent $n \times n$ matrices. The following result of Lusztig [Lus] gives a cell decomposition of $U_{\geq 0}^{\mathrm{fin}}$.

Theorem 1.1.

(1) *We have*
$$U_{\geq 0}^{\mathrm{fin}} = \bigsqcup_{w \in S_n} U_{\geq 0}^{w}$$
where
$$U_{\geq 0}^{w} = \{e_{i_1}(a_1)e_{i_2}(a_2)\cdots e_{i_\ell}(a_\ell) \mid a_k > 0\}$$
and $w = s_{i_1}s_{i_2}\cdots s_{i_\ell}$ is a reduced expression.

(2) *The set $U_{\geq 0}^{w}$ does not depend on the choice of reduced expression, and the map $e_{\mathbf{i}} : \mathbb{R}_{>0}^{\ell} \mapsto U_{\geq 0}^{w}$ is a bijection.*

The cells $U_{\geq 0}^{w}$ can also be obtained by intersecting $U_{\geq 0}^{\mathrm{fin}}$ with the Bruhat decomposition $GL_n(\mathbb{R}) = \sqcup_{w \in S_n} B_- w B_-$. In Theorem 3.2 we establish the analogue of Theorem 1.1 for the totally nonnegative part $U_{\geq 0}^{\mathrm{pol}}$ of the polynomial loop group, with the affine symmetric group replacing the symmetric group.

1.2. **Infinite products of Chevalley generators.** Let $\tilde{W}$ denote the affine symmetric group $\tilde{S}_n$, with simple generators $\{s_i \mid i \in \mathbb{Z}/n\mathbb{Z}\}$. An infinite word $\mathbf{i} = i_1 i_2 \cdots$ in the alphabet $\mathbb{Z}/n\mathbb{Z}$ is *reduced* if each initial subword is a reduced word for some element of $\tilde{W}$.

For infinite reduced words the map $e_{\mathbf{i}}$ of (1) does not satisfy many of the good properties which exist for finite reduced words. Let $E_{\mathbf{i}}$ denote the image of $e_{\mathbf{i}}$ and let $\Omega = \cup_{\mathbf{i}} E_{\mathbf{i}}$. Then in contrast to the finite case,

(1) The map $e_{\mathbf{i}}$ is not injective in general.
(2) We can find infinite reduced words $\mathbf{i}, \mathbf{j}$ such that $E_{\mathbf{i}} \subsetneq E_{\mathbf{j}}$.
(3) We can find infinite reduced words $\mathbf{i}, \mathbf{j}$ such that $E_{\mathbf{i}} \cap E_{\mathbf{j}} \neq \emptyset$ but neither $E_{\mathbf{i}}$ nor $E_{\mathbf{j}}$ is contained in the other.
(4) Ω is not a semigroup.

We shall tackle these difficulties by:

(1) Giving a conjectural classification (Conjecture 6.3) of infinite reduced words $\mathbf{i}$ such that $e_{\mathbf{i}}$ is injective, and proving this in an important case when the domain is restricted (Proposition 6.9).
(2) Giving a criterion (Theorem 5.5) for $E_{\mathbf{i}} \subset E_{\mathbf{j}}$, using the notion of *braid limits* and the *limit weak order*.
(3) Showing that the union of $E_{\mathbf{i}}$ over the finite set of infinite Coxeter elements covers Ω (Corollary 5.8).
(4) For each $X \in \Omega$, constructing a distinguished factorization $X = e_{\mathbf{i}}(\mathbf{a})$ (Theorem 7.5).

(5) Showing that the limit semigroup generated by Ω satisfies some form of unique factorization (Theorem 7.11).

1.3. **Limit weak order.** The inversion set $\mathrm{Inv}(\mathbf{i})$ of an infinite reduced word is an infinite set of positive real roots of $\tilde{W}$. These inversion sets were classified by Cellini and Papi [CP] (who called them compatible sets) and by Ito [Ito] (who called them biconvex sets).

Inclusion of inversion sets gives rise to a partial order on (equivalence classes of) infinite reduced words, which we call the *limit weak order*. We show that the limit weak order $(\tilde{\mathcal{W}}, \leq)$ can also be obtained by performing (possibly infinite) sequences of braid moves on infinite reduced words. We encourage the reader to look at Example 4.2 for an example of such a sequence, which we call a *braid limit*, denoted $\mathbf{i} \to \mathbf{j}$.

The limit weak order $(\tilde{\mathcal{W}}, \leq)$ is an infinite poset which (unlike usual weak order) contains intervals which themselves are infinite. To analyze it, we divide $\tilde{\mathcal{W}}$ into *blocks*. Each block is isomorphic to a product of usual weak orders of (smaller) affine symmetric groups (Theorem 4.12). The partial order between the blocks themselves is isomorphic to the face poset of the braid arrangement (Theorem 4.11). We explicitly express (Proposition 4.13) the (unique) minimal element of each block as an infinite reduced word. In particular, the minimal elements of $(\tilde{\mathcal{W}}, \leq)$ are exactly the infinite products c^∞ of Coxeter elements c of $\tilde{W}$ (Theorem 4.17), which are in bijection with the edges of the braid arrangement.

Many of the results concerning limit weak order generalize to other infinite Coxeter groups, but some (for example, Theorem 4.17) do not.

1.4. **Braid limits and total nonnegativity.** When we perform infinitely many braid transformations to a product $e_{i_1}(a_1)e_{i_2}(a_2)\cdots$, and take a limit, a priori it is not clear that the resulting product is equal to the original one. In fact, this is false in Kac-Moody generality. The following central result (Theorem 5.5) shows that this is true in affine type A, thus laying a foundation for our investigations of Ω.

Theorem (TNN braid limit theorem). *If $\mathbf{i} \to \mathbf{j}$ is a braid limit between two infinite reduced words, then $E_{\mathbf{i}} \subset E_{\mathbf{j}}$. In other words, every totally nonnegative matrix X which can be expressed as $X = e_{\mathbf{i}}(\mathbf{a})$ can be expressed as $X = e_{\mathbf{j}}(\mathbf{a}')$.*

As stated above, the infinite Coxeter elements c^∞ are the minimal elements of limit weak order. It follows that we have the finite (but not disjoint) union $\Omega = \cup_c E_{c^\infty}$. We use the TNN braid limit theorem to show that $e_{\mathbf{i}}$ can only be injective when $\mathbf{i}$ is minimal in its block of limit weak order (Proposition 6.2). We conjecture that the converse also holds (Conjecture 6.3). Finally we use the ϵ-sequence of [LPI] to establish injectivity in some cases (Proposition 6.9).

1.5. **ASW factorizations.** To tackle the lack of injectivity of $e_{\mathbf{i}}$, we give two different approaches.

In the first approach, we study the *ASW factorization* of [LPI], applied to matrices $X \in \Omega$. Let us recall the definition here. For $X \in U_{\geq 0}$ let $\epsilon_i(X) = \lim_{j\to\infty} \frac{x_{i,j}}{x_{i+1,j}}$. It was shown in [LPI, Lemma 5.3] that there exists a factorization $X = N(\epsilon_1, \ldots, \epsilon_n)Y$, where $N(\epsilon_1, \ldots, \epsilon_n)$ is a curl with parameters $\epsilon_i = \epsilon_i(X)$ and $Y \in U_{\geq 0}$ is some totally nonnegative matrix. We refer to the extraction of the curl factor $N(\epsilon_1, \ldots, \epsilon_n)$ from X as the ASW factorization of X. We also use the same terminology for the *repeated* extraction of such a factor.

The main difficulty here can be stated rather simply: suppose $X \in \Omega$ and $X = e_i(a)X'$, where $a > 0$ and $X' \in U_{\geq 0}$, then is X' necessarily in Ω? We answer this affirmatively. As a consequence, we obtain a distinguished factorization of each $X \in \Omega$, decomposing Ω into a *disjoint* union of pieces which we call *ASW-cells* (Theorem 7.5 and Proposition 7.15). The ASW-cells are labeled by certain pairs $(w, v) \in \tilde{W} \times \tilde{W}$ of affine permutations, which we call compatible. Our study of ASW factorization also leads to our first proof of the TNN braid limit theorem, and in addition proves the following theorem (Theorem 7.12).

Theorem (Unique factorization in $\mathbb{L}_r$)**.** *Denote by $\mathbb{L}_r$ the right limit-semigroup generated by Chevalley generators (see Section 7.3 for precise definitions). Each element of $\mathbb{L}_r$ has a unique factorization into factors which lie in Ω, with possibly one factor which is a finite product of Chevalley generators.*

1.6. **Greedy factorizations.** In a second approach to the lack of injectivity of $e_{\mathbf{i}}$, we study *greedy* factorizations. These are factorizations $X = e_{\mathbf{i}}(\mathbf{a})$ where for a fixed $\mathbf{i}$, a_1 is maximal and having factored out $e_{i_1}(a_1)$, the second parameter a_2 is also maximal, and so on. Clearly, if $\mathbf{i}$ is fixed, there is at most one greedy factorization of X, so "injectivity" is automatic. Our main result (Theorem 9.6) concerning greedy factorizations is that they are preserved under braid moves (or even braid limits). We also give formulae in some special cases for the parameters $a_1, a_2, \ldots$ in a greedy factorization in terms of limits of ratios of minors of X. We have already studied minor limit ratios in [LPI]. The minor limit ratios used for greedy factorizations are distinguished by the fact that a single limit involves ratios of minors of different sizes.

1.7. **Totally positive exchange lemma.** One of our proofs of the TNN braid limit theorem is based upon the Totally Positive Exchange Lemma (Theorem 8.1). This is a result about *finite* products $e_{i_1}(a_1) \cdots e_{i_k}(a_k)$ of Chevalley generators, which seems to be of independent interest. Recall the usual exchange condition for Coxeter groups.

Theorem (Exchange Lemma)**.** *If $\bar{w} = s_{i_1} s_{i_2} \ldots s_{i_k}$ is a reduced expression for an element w of a Coxeter group, and $s_r \bar{w}$ is not reduced, then $s_r w = s_{i_1} \ldots \hat{s}_{i_l} \ldots s_{i_k}$ for a unique index l, where $\hat{s}_{i_l}$ denotes omission of a generator.*

The Totally Positive Exchange Lemma states that in (affine) type A when an exchange is performed on the level of Chevalley generators, certain inequalities between the parameters a_i before and after the exchange hold.

Theorem (Totally Positive Exchange Lemma)**.** *Suppose*

$$X = e_r(a) e_{i_1}(a_1) \cdots e_{i_\ell}(a_\ell) = e_{i_1}(a'_1) \cdots e_{i_\ell}(a'_\ell) e_j(a')$$

are reduced products of Chevalley generators such that all parameters are positive (so that j has been exchanged for r). For each $m \leq \ell$ and each $x \in \mathbb{Z}/n\mathbb{Z}$ define $S = \{s \leq m \mid i_s = x\}$. Then

$$\sum_{s \in S} a'_{i_s} \leq \begin{cases} \sum_{s\in S} a_{i_s} & \text{if } x \neq r, \\ a + \sum_{s \in S} a_{i_s} & \text{if } x = r. \end{cases}$$

We give two proofs of the TP Exchange Lemma. The first proof relies on explicit formulae for the parameters a_i, given by the Berenstein-Zelevinsky Chamber Ansatz [BZ]. The second proof is less direct, and relies on reducing the result to a statement about

calculating certain joins in weak order. We shall return to the TP Exchange Lemma in a more general setting in future work [LPKM].

1.8. **Open problems and conjectures.** Section 10 contains a list of questions and conjectures together with some partial results, most of which are concerned with the subsets $E_{\mathbf{i}} \subset \Omega$ and the maps $e_{\mathbf{i}}$. A particularly powerful conjecture is the Principal ideal conjecture (Conjecture 10.6) which states that the set $\{\mathbf{i} \mid X \in E_{\mathbf{i}}\}$ is a principal ideal in limit weak order.

2. Notations and definitions

2.1. **Total nonnegativity.** To every $\bar{X}(t) \in GL_n(\mathbb{R}((t)))$ we associate an infinite periodic matrix X, which are related via

$$\bar{x}_{ij}(t) = \sum_{k=-\infty}^{\infty} x_{i,j+kn} t^k.$$

As in [LPI], we will abuse notation by rarely distinguishing between a matrix $\bar{X}(t) \in GL_n(\mathbb{R}((t)))$, and its unfolding X which is an infinite periodic matrix. The matrix $\bar{X}(t)$ is called the folded matrix. For finite sets $I, J \subset \mathbb{Z}$ of the same cardinality we let $\Delta_{I,J}(X)$ denote the corresponding minor, always in the unfolded matrix.

Following [LPI], we let $U \subset GL_n(\mathbb{R}((t)))$ denote the group of unipotent upper-triangular matrices, $U_{\geq 0}$ (resp. $U_{>0}$) denote the totally nonnegative (resp. totally positive) part of U. Both $U_{\geq 0}$ and $U_{>0}$ are semigroups. We let $U^{\mathrm{pol}} \subset U$ and $U_{\geq 0}^{\mathrm{pol}} \subset U_{\geq 0}$ denote the corresponding matrices which belong to the polynomial loop group (both X and X^{-1} are required to have polynomial entries).

Recall that in [LPI] we have defined an (anti-)involution $X \mapsto X^{-c}$ of $U_{\geq 0}$. We say that $X \in U$ is *entire* if all the (folded) entries are entire functions. We say that $X \in U$ is *doubly-entire* if both X and X^{-c} is entire. We say that X is *finitely-supported* if all the (folded) entries are polynomials.

Let $I = \{i_1 < i_2 < \cdots < i_k\}$ and $J = \{j_1 < j_2 < \cdots < j_k\}$ be subsets of $\mathbb{Z}$. Write $I \leq J$ if $i_r \leq j_r$ for $r \in [1, k]$. The minors $\Delta_{I,J}(X)$ are the upper-triangular minors: all other minors vanish on $U_{\geq 0}$. We say that $X \in U_{\geq 0}$ is *totally positive* (see [LPI, Corollary 5.9]) if for all $I \leq J$, the minor $\Delta_{I,J}(X)$ is strictly positive.

2.2. **Affine symmetric group.** Let $\tilde{W}$ denote the affine symmetric group, with simple generators $\{s_i \mid i \in \mathbb{Z}/n\mathbb{Z}\}$, satisfying the relations $s_i^2 = 1$, $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ and $s_i s_j = s_j s_i$ for $|i - j| > 1$. The indices are always to be taken modulo n. The length $\ell(w)$ for $w \in \tilde{W}$ is the length of the shortest expression $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ of w in terms of simple generators. We call such an expression a reduced expression, and $i_1 i_2 \cdots i_\ell$ a reduced word for w. The (right) weak order on $\tilde{W}$ is defined by $v < w$ if $w = vu$ for u satisfying $\ell(w) = \ell(v) + \ell(u)$. Right weak order is graded by the length function $\ell(w)$, and the covering relations are of the form $w < ws_i$. A left descent of $w \in \tilde{W}$ is an index (or simple root, or simple generator) i (or α_i or s_i) such that $s_i w < w$. Similarly one defines ascents. Note that i is a left descent of w if and only if there is a reduced word for w beginning with i.

The affine symmetric group $\tilde{W}$ can be identified with the group of bijections $w : \mathbb{Z} \to \mathbb{Z}$ satisfying $w(i+n) = w(i) + n$ and $\sum_{i=1}^{n} w(i) = n(n+1)/2$. Group multiplication is given

by function composition. Left multiplication by s_i swaps the values i and $i+1$, while right multiplication swaps positions. The window notation for $w \in \tilde{W}$ is the sequence $[w(1)w(2)\cdots w(n)]$, which completely determines w. The symmetric group $W = S_n$ embeds in $\tilde{W}$ in the obvious manner.

Let $Q^\vee = \oplus_{i=1}^{n-1} \mathbb{Z} \cdot \alpha_i^\vee$ denote the (finite) coroot lattice, which we identify with

$$\{(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n \mid \sum_{i=1}^n \lambda_i = 0\}.$$

Given $\lambda \in Q^\vee$, one has a translation element $t_\lambda \in \tilde{W}$, given by

$$t_\lambda(i) = i + n\,\lambda_i,$$

so that $t_{(0,0,\ldots,0)}$ is the identity affine permutation. The affine symmetric group can be presented as a semidirect product $W \ltimes Q^\vee$, where $vt_\lambda v^{-1} = t_{v\cdot\lambda}$ for $v \in W$. We say that $\lambda \in Q^\vee$ is *dominant* if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

Lemma 2.1. *Let $\lambda \in Q^\vee$ be dominant and $w \in W$ be arbitrary. Then*

$$\ell(t_{w\cdot\lambda}) = 2(n\lambda_1 + (n-1)\lambda_2 + \cdots + \lambda_{n-1}).$$

In particular, $\ell(t_{w\cdot\lambda})$ does not depend on w.

Let $\Delta_0 = \{\alpha_{i,j} \mid 1 \leq i \neq j \leq n\}$ denote the root system of W, and write $\alpha_i = \alpha_{i,i+1}$ for the simple roots. We let Δ denote the root system of $\tilde{W}$, with simple roots $\{\alpha_i \mid i \in \mathbb{Z}/n\mathbb{Z}\}$ and null root $\delta = \alpha_0 + \alpha_1 + \cdots + \alpha_{n-1}$. We have $\Delta = \{n\delta \mid n \in \mathbb{Z} - \{0\}\} \cup \{n\delta + \alpha \mid n \in \mathbb{Z}$ and $\alpha \in \Delta_0\}$. The real roots $\{n\delta + \alpha \mid n \in \mathbb{Z}$ and $\alpha \in \Delta\}$ are denoted Δ_{re}.

Recall that we have

$$\Delta_0 = \Delta_0^+ \cup \Delta_0^- = \{\alpha_{i,j} = \alpha_i + \cdots + \alpha_{j-1} \mid i < j\} \cup \{\alpha_{i,j} = -\alpha_{j,i} \mid i > j\}.$$

Thus the root $\alpha_{i,j}$ is positive if $i < j$, negative if $i > j$, and positive simple if $j = i+1$. The roots in Δ_{re} are $\alpha_{i,j} + k\delta$, $k \in \mathbb{Z}$. A real affine root is positive if $i < j$ and $k \geq 0$, or if $i > j$ and $k \geq 1$.

3. Polynomial loop group

3.1. **Relations for Chevalley generators.** By [LPI, Theorem 2.6] the semigroup $U_{\geq 0}^{\mathrm{pol}}$ is generated by the Chevalley generators $e_i(a)$ with nonnegative parameters $a \geq 0$. We recall the standard relations for Chevalley generators [Lus]:

$$e_i(a)\,e_j(b) = e_j(b)e_i(a) \qquad \text{if } |i-j| \geq 2 \tag{2}$$

$$e_i(a)\,e_{i+1}(b)\,e_i(c) = e_{i+1}(bc/(a+c))\,e_i(a+c)\,e_{i+1}(ab/(a+c)) \quad \text{for each } i \in \mathbb{Z}/n\mathbb{Z} \tag{3}$$

for nonnegative parameters a, b, c. For a reduced word $\mathbf{i} = i_1 i_2 \cdots i_\ell$ of $w \in \tilde{W}$, and a collection of parameters $a_k \in \mathbb{R}$, write $e_{\mathbf{i}}(\mathbf{a})$ for $e_{i_1}(a_1)\cdots e_{i_\ell}(a_\ell)$. Denote $E_{\mathbf{i}}$ the image of the map $\mathbf{a} \mapsto e_{\mathbf{i}}(\mathbf{a})$, as $\mathbf{a}$ ranges over $\mathbb{R}^\ell_{>0}$. The following result follows from relations (2) and (3).

Lemma 3.1. *If $\mathbf{i}$ and $\mathbf{j}$ are two reduced words of $w \in \tilde{W}$ then $E_{\mathbf{i}} = E_{\mathbf{j}}$.*

Therefore we can introduce the notation $E_w = E_{\mathbf{i}}$ which is independent of the reduced word $\mathbf{i}$ of w.

3.2. **Decomposition of $U_{\geq 0}^{\mathrm{pol}}$.**

Theorem 3.2. *We have a disjoint union*

$$U_{\geq 0}^{\mathrm{pol}} = \bigsqcup_{w \in \tilde{W}} E_w.$$

The fact that the subsets $\{E_w\}_{w\in\tilde{W}}$ cover $U_{\geq 0}^{\mathrm{pol}}$ follows from [LPI, Theorem 2.6]. To prove that the E_w-s are disjoint we will describe a necessary condition for $X \in U_{\geq 0}^{\mathrm{pol}}$ to belong to E_w.

We refer to the matrix entry positions of an infinite $(\mathbb{Z}\times\mathbb{Z})$ matrix X as *cells*. Define a partial order on cells: $(i,j) \leq (i',j')$ if $i \geq i'$ and $j \leq j'$. In other words $c \leq c'$ if c' is to the northeast of c. A cell with coordinates $(i, w^{-1}(i))$ for some i is called a w-*dot*. The collection of w-dots is denoted C_w. We say that a finite set of cells C is w-*dominated* if for every cell (i,j) we have

$$\#\{c \in C \mid c \geq (i,j)\} \leq \#\{c \in C_w \mid c \geq (i,j)\}.$$

Let $I = \{i_1 < i_2 < \cdots < i_k\}$, $J = \{j_1 < j_2 < \cdots j_k\} \subset \mathbb{Z}$. Define $C = C(I,J)$ to be the set of cells $\{(i_1,j_1),(i_2,j_2),\ldots,(i_k,j_k)\}$.

Proposition 3.3. *Let $w \in \tilde{W}$ and suppose $X \in E_w$. Let $I \leq J$. Then $\Delta_{I,J}(X) > 0$ if and only if $C(I,J)$ is w-dominated.*

Proof. We shall write $\Delta_C(X)$ for $\Delta_{I,J}(X)$ when $C = C(I,J)$.

The proof proceeds by induction on the length $\ell(w)$ of w. The base case of the identity permutation $w = \mathrm{id}$ is trivial: an upper-triangular minor is non-zero if and only if all cells in $C(I,J)$ are on the diagonal, if and only if $C(I,J)$ is id-dominated.

Assume now that $w = s_i v$ where $\ell(w) = \ell(v)+1$, and that we already know the validity of the statement for elements of E_v. The set C_w differs from C_v by swapping the cells in the $kn+i$-th and $kn+i+1$-th rows for each k. It follows from $\ell(w) = \ell(v)+1$ that any v-dominated set is also w-dominated. Multiplication of $Y \in E_v$ by $e_i(a)$ on the left adds a times row $kn+i+1$ to the row $kn+i$ for each $k \in \mathbb{Z}$. In particular, any positive minor of Y is also a positive minor of X.

Now let $C = C(I,J)$ be w-dominated for some $I \leq J$. We may assume that $C(I,J)$ does not contain any cells on the diagonal, since the value of $\Delta_{I,J}(X)$ does not change if the diagonal cells are removed; in addition w-dominance is preserved under removal of cells. If C is also v-dominated then $\Delta_C(X) \geq \Delta_C(Y) > 0$ by the inductive assumption. Otherwise C is not v-dominated. Since C_w and C_v differ only in the $(kn+i)$-th and $(kn+i+1)$-th rows, C must contain a cell in one of those rows. Checking a number of cases, one deduces that for some k, C contains a cell in the $(kn+i)$-th row but not in the $(kn+i+1)$-th row. Let C' be obtained from C by moving all cells in the $kn+i$-th rows down one row, whenever the row below is not occupied. It is easy to check that C' is v-dominated. Furthermore, $\Delta_C(X)$ is a positive linear combination of minors of Y, one of which is $\Delta_{C'}(Y)$. Thus $\Delta_C(X) > 0$ if C is w-dominated.

Now suppose $C = C(I,J)$ is not w-dominated for some $I \leq J$. The minor $\Delta_C(X)$ is a linear combination of minors of the form $\Delta_{C'}(Y)$, where C' is obtained from C by moving cells in the $kn+i$-th rows for some values of k down one row (assuming the row below is not occupied). We claim that all the minors $\Delta_{C'}(Y)$ vanish. It is enough to show that C' is never v-dominated, assuming that C' consists only of cells above the diagonal.

For each (a,b), let $A(a,b) = \#\{c \in C \mid c \geq (a,b)\}$, $A'(a,b) = \#\{c \in C' \mid c \geq (a,b)\}$, $A_w(a,b) = \#\{c \in C_w \mid c \geq (a,b)\}$, and $A_v(a,b) = \#\{c \in C_v \mid c \geq (a,b)\}$. Suppose (a,b) satisfies $A(a,b) > A_w(a,b)$. If a is not of the form $kn+i$ then we have $A(a,b) = A'(a,b)$ and $A_w(a,b) = A_v(a,b)$ so that C' is not v-dominated. So assume $a = kn+i$. We may assume that C contains a cell $c = (a,r)$ in row a, and that $b \leq r$. If $b \leq w^{-1}(a)$ then $A(a-1,b) > A_w(a-1,b)$, reducing to the previous case. If $b > w^{-1}(a)$ then $A_v(a+1,b) = A_w(a,b)$ and $A'(a+1,b) \geq A(a,b)$, which implies C' is not v-dominated. □

Example 3.1. Let $n = 3$ and let $w = s_0 s_1$. The window notation for w is $[2,0,4]$ and the w-dots are the cells with coordinates $(3k+1, 3k)$, $(3k, 3k+2)$ and $(3k+2, 3k+1)$ for $k \in \mathbb{Z}$. Take $I = (0,1)$ and $J = (1,2)$. Then $I \leq J$, but $C(I,J)$ is not w-dominated. Indeed, $C(I,J) = \{(0,1),(1,2)\}$ and for $(i,j) = (1,1)$ we have $\#\{c \in C(I,J) \mid c \geq (1,1)\} = 2$, while $\#\{c \in C_w \mid c \geq (1,1)\} = 1$. Therefore for $X \in E_w$ we have $\Delta_{I,J}(X) = 0$. On the other hand, if we pick $I = (-2,0,1)$ and $J = (-1,0,2)$ then it is not hard to check that $C(I,J)$ is w-dominated and therefore $\Delta_{I,J}(X) > 0$ for $X \in E_w$.

Remark 3.2. Proposition 3.3 can be applied in the special case $w \in W$, naturally generalizing the conditions appearing in [FZ, Proposition 4.1]. Note however that unlike [FZ] we deal only with totally nonnegative matrices and do not aim to provide a minimal set of sufficient conditions.

Proof of Theorem 3.2. We claim that the minor vanishing/non-vanishing conditions of Proposition 3.3 are incompatible for two distinct elements $w, v \in \tilde{W}$. Indeed, assume there exists $X \in E_w \cap E_v$. The set of numbers $A_w(i,j) = \#\{c \in C_w \mid c \geq (i,j)\}$ for all i,j completely determine w. If $w \neq v$ there is (i,j) such that $A_w(i,j) \neq A_v(i,j)$ and $A_w(i',j') = A_v(i',j')$ for all $(i',j') > (i,j)$. We may assume that $A_w(i,j) > A_v(i,j)$. The w-dots and v-dots strictly to the north or east or northeast of (i,j) coincide. Let I (resp. J) be the rows (resp. columns) containing w-dots to northeast of (i,j), including (i,j) itself. The fact that $I \leq J$ is easy to see by induction. It is clear that $C = C(I,J)$ is not v-dominated, and one checks a number of cases to see that C is w-dominated. We obtain a contradiction from Proposition 3.3 by looking at $\Delta_C(X)$. □

The following result is crucial for later parts of the paper.

Theorem 3.4.

1. *For* $\mathbf{i}$ *a reduced decomposition of* w*, the map* $e_{\mathbf{i}} : \mathbb{R}^\ell_{>0} \to E_w$ *is injective.*
2. *If* $X \in E_w$ *and* $X = YZ$ *where* Y *and* Z *are totally nonnegative, then* $Y \in E_v$ *and* $Z \in E_u$ *for some* $v, u \in \tilde{W}$ *and* $v \leq w$ *in weak order.*

Proof. We prove (1). Assume that $\mathbf{a}, \mathbf{a}' \in \mathbb{R}^\ell_{>0}$ are two distinct sets of parameters such that $e_{\mathbf{i}}(\mathbf{a}) = X = e_{\mathbf{i}}(\mathbf{a}')$. Without loss of generality we assume $a_1 \neq a'_1$, for otherwise we may remove $e_{i_1}(a_1)$ from X and remove s_{i_1} from w, and apply the argument to the resulting affine permutation. We also assume without loss of generality that $a_1 > a'_1$. Then $e_{i_1}(-a'_1)X$ lies both in E_w and $E_{s_i w}$, which is a contradiction to Theorem 3.2.

We prove (2). Since X is finitely supported, so are Y and Z. By [LPI, Lemma 5.1 and Theorem 5.5] and the calculation $1 = \det(\bar{X}(t)) = \det(\bar{Y}(t))\det(\bar{Z}(t))$, it follows that Y and Z factor into a product of $e_i(a)$-s, and thus for some $v, u \in \tilde{W}$ we have $Y \in E_v$, $Z \in E_u$. Finally, to see why $v \leq w$ one can think of multiplying Y by a sequence of Chevalley generators from Z to obtain matrices in $E_v, E_{v^{(1)}}, E_{v^{(2)}}, \ldots, E_w$. It is easy to see that $v \leq v^{(1)} \leq v^{(2)} \leq \cdots \leq w$. □

If $\mathbf{i} = i_1 i_2 \cdots i_\ell$ and $\mathbf{j} = j_1 j_2 \cdots j_\ell$ are two reduced words for $w \in \tilde{W}$, then applying the relations (2) and (3), we obtain a map

$$R_{\mathbf{i}}^{\mathbf{j}} : \mathbb{R}_{>0}^{\ell} \to \mathbb{R}_{>0}^{\ell}$$

such that $e_{\mathbf{i}}(\mathbf{a}) = e_{\mathbf{j}}(R_{\mathbf{i}}^{\mathbf{j}}(\mathbf{a}))$. The following is an immediate corollary of Theorem 3.4.

Corollary 3.5. *The map $R_{\mathbf{i}}^{\mathbf{j}}$ is well-defined, and does not depend on the order in which we apply* (2) *and* (3).

4. Infinite reduced words, and braid limits

4.1. **Biconvex sets.** Let $I \subset \Delta_{\mathrm{re}}^+$ be a (possibly infinite) set of positive real affine roots. We call I *biconvex* if for any $\alpha, \beta \in \Delta^+$ one has

(1) if $\alpha, \beta \in I$ and $\alpha + \beta \in \Delta$ then $\alpha + \beta \in I$,
(2) if $\alpha + \beta \in I$ then either $\alpha \in I$ or $\beta \in I$.

Note that in (1) one must include the case that $\alpha + \beta$ is not a real root. Biconvex sets were studied in [CP, Ito] for an arbitrary affine Weyl group. (Cellini and Papi [CP] use the word "compatible" instead.)

Let $I \subset \Delta_{\mathrm{re}}^+$ be a biconvex set. It is easy to see that for each $\alpha \in \Delta_0^+$, the intersection

$$I \cap \{\ldots, 3\delta - \alpha, 2\delta - \alpha, \delta - \alpha, \alpha, \alpha + \delta, \alpha + 2\delta, \ldots\}$$

is one of the following: (a) empty, (b) $\{\alpha, \alpha + \delta, \ldots, \alpha + m_\alpha \delta\}$, (c) $\{\alpha, \alpha + \delta, \ldots\}$, (d) $\{\delta - \alpha, 2\delta - \alpha, \ldots, -m_\alpha \delta - \alpha\}$, or (e) $\{\delta - \alpha, 2\delta - \alpha, \ldots\}$. In (b), $m_\alpha > 0$ but in (d), $m_\alpha < 0$. In cases (a), (c), (e), we set m_α to be 0, ∞, $-\infty$ respectively. The proof of the following result is straightforward.

Proposition 4.1. *A set of positive real roots is biconvex if and only if for any $\alpha, \beta, \gamma \in \Delta_0^+$ such that $\alpha + \beta = \gamma$ we have one of the following possibilities for m_α, m_β and m_γ:*

m_α	m_β	m_γ
finite	*finite*	$m_\alpha + m_\beta$
finite	*finite*	$m_\alpha + m_\beta - 1$
$\pm\infty$	*finite*	$\pm\infty$
finite	$\pm\infty$	$\pm\infty$
$\pm\infty$	$\pm\infty$	$\pm\infty$
$\pm\infty$	$\mp\infty$	*anything*

4.2. **Infinite reduced words, inversion sets.** If $w \in \tilde{W}$ has reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_k}$, then the *inversion set* of w is the set of real roots given by

$$\mathrm{Inv}(w) = \{\alpha_{i_1}, s_{i_1}\alpha_{i_2}, s_{i_1}s_{i_2}\alpha_{i_3}, \ldots, s_{i_1}s_{i_2}\cdots s_{i_{k-1}}\alpha_{i_k}\} \subset \Delta_{\mathrm{re}}.$$

It is well known that $|\mathrm{Inv}(w)| = \ell(w)$.

The inversions can be read directly from (the window notation of) an affine permutation $w \in \tilde{W}$ as follows. For a finite positive root $\alpha = \alpha_{i,j} \in \Delta_0^+$ let

$$m_\alpha = \min\{k \mid nk > w^{-1}(i) - w^{-1}(j)\}.$$

Then if $m_\alpha > 0$ the affine roots $\alpha, \ldots, \alpha + (m_\alpha - 1)\delta$ are inversions of w, while if $m_\alpha < 0$ the affine roots $\delta - \alpha, \ldots, (-m_\alpha)\delta - \alpha$ are inversions of w. If $m_\alpha = 0$ neither α nor $-\alpha$ are inversions of w. In particular, if $\alpha + m\delta$ (resp. $m\delta - \alpha$) is an inversion for $m > 0$, then so is $\alpha + m'\delta$ (resp. $m'\delta - \alpha$) for $0 \leq m' \leq m$.

Let $\mathbf{i} = i_1 i_2 i_3 \cdots$ be either a finite, or (countably) infinite word with letters from $\mathbb{Z}/n\mathbb{Z}$. We call $\mathbf{i}$ reduced if $w_{\mathbf{i}}^{(k)} = s_{i_1} s_{i_2} \cdots s_{i_k} \in \tilde{W}$ has length k for every k. We define the inversion set of $\mathbf{i}$ to be $\mathrm{Inv}(\mathbf{i}) = \cup_k \mathrm{Inv}(w_{\mathbf{i}}^{(k)}) \subset \Delta_{\mathrm{re}}^+$. We call a subset $I \subset \Delta_{\mathrm{re}}^+$ an inversion set if $I = \mathrm{Inv}(\mathbf{i})$ for some finite or infinite reduced word $\mathbf{i}$. If $w \in \tilde{W}$ then by w^∞ we mean the infinite word obtained by repeating a reduced word for w. By Lemma 2.1, t^∞ is reduced for any translation. If $\mathbf{i}$ is an infinite word, and $w \in \tilde{W}$ we may write $w\mathbf{i}$ for the infinite word obtained by prepending $\mathbf{i}$ with a reduced word of w. Note that if $w\mathbf{i}$ is reduced then

$$\mathrm{Inv}(w\mathbf{i}) = \mathrm{Inv}(w) \sqcup w \cdot \mathrm{Inv}(\mathbf{i}) \tag{4}$$

Biconvex sets were studied and classified in the case of an arbitrary affine Weyl group by Ito [Ito], and Cellini and Papi [CP] (under the name of compatible sets).

Theorem 4.2 ([Ito, CP])**.** *Suppose $I \subset \Delta_{\mathrm{re}}^+$ is infinite. Then the following are equivalent:*

(1) *I is an inversion set;*
(2) *I is biconvex;*
(3) *$I = \mathrm{Inv}(vt^\infty) = \mathrm{Inv}(v) \sqcup v \cdot \mathrm{Inv}(t^\infty)$ for some $v \in \tilde{W}$ and translation element t such that vt^∞ is reduced.*

For completeness, we provide a proof of Theorem 4.2.

Proof. It is well known, and easy to prove by induction, that $\mathrm{Inv}(w)$ is biconvex for $w \in \tilde{W}$. Since increasing unions of biconvex sets are biconvex, we have (1) implies (2). Since (3) implies (1) is obvious, it suffices to show that every infinite biconvex set I is of the form $\mathrm{Inv}(vt^\infty)$. Let $\{m_\alpha \mid \alpha \in \Delta_0^+\}$ be as in Proposition 4.1. We claim that the m_α-s can be all made into $\{0, +\infty, -\infty\}$, in finitely many steps, by a sequence of the following operations on I: take some $\alpha_i \in I$ for $i \in \mathbb{Z}/n\mathbb{Z}$ (the root $\alpha_0 = \delta - \alpha_{1,n}$ is allowed), then change I to $s_i \cdot (I - \{\alpha_i\})$. From the definitions, one sees that $s_i \cdot (I - \{\alpha_i\})$ is still biconvex, and that in this way the "finite" m_α-s can be made closer to 0. This sequence of operations corresponds to the $v \in \tilde{W}$ of (3). To complete the proof, Proposition 4.3 below shows that if every $m_\alpha \in \{0, +\infty, -\infty\}$ then I is of the form $\mathrm{Inv}(t_\lambda^\infty)$ for some $\lambda \in Q^\vee$. ☐

4.3. **Blocks and the braid arrangement.** The braid arrangement is the finite, central hyperplane arrangement in $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}\}$ given by the hyperplanes $x_i - x_j = 0$, for $i < j$. Equivalently, the hyperplanes may be written as $\langle \alpha_{i,j}, x \rangle = 0$.

A *pre-order* $\preceq$ on a set S is a reflexive, transitive relation. Any pre-order determines an equivalence relation: $s \sim s'$ if $s \preceq s'$ and $s' \preceq s$. A pre-order is called *total* if the induced partial order on equivalence classes is a total order, or equivalently, if every pair of elements can be compared. The faces of the braid arrangement are in bijection with total pre-orders on $[n]$. Total pre-orders are essentially the same as set compositions. For example, the pre-order $\{2,4\} \prec \{1,5\} \prec \{3\}$ corresponds to the set composition $\Gamma = (\{2,4\}, \{1,5\}, \{3\})$, which corresponds to the open face $F = \{(x_1, x_2, x_3, x_4, x_5) \mid x_2 = x_4 < x_1 = x_5 < x_3\}$ of the braid arrangement.

We say that two infinite biconvex sets I and J are *in the same block* if $|I - J| + |J - I|$ is finite.

Proposition 4.3. *The map $F \mapsto \mathrm{Inv}(t_\lambda^\infty)$ establishes a bijection between the (open) faces of the braid arrangement (excluding the lowest dimensional face) and blocks of infinite biconvex sets, where λ is any element of $Q^\vee \cap F$.*

It is clear that a block B is determined uniquely by knowing which m_α-s are infinite, and among those which are $+\infty$ and which are $-\infty$. Using the m_α-s, we define a relation $\preceq_B$ on $[n]$ as follows: if $i < j$ in $[n]$ then $i \preceq j$ if $m_{\alpha_{i,j}}$ is finite or $+\infty$, and $j \preceq i$ if $m_{\alpha_{i,j}}$ is finite or $-\infty$.

Lemma 4.4. *The relation $\preceq_B$ defined above is a total pre-order.*

Proof. It suffices to show that $\preceq$ is transitive. Suppose $i \preceq j$ and $j \preceq k$. There are several cases to consider. We take for example the case $i < k < j$. In that case we have $m_{\alpha_{i,j}}$ is finite or $+\infty$ and $m_{\alpha_{k,j}}$ is finite or $-\infty$. Then looking through the table in Proposition 4.1, we deduce that $m_{\alpha_{i,k}}$ is either finite or $+\infty$, which implies $i \preceq k$. The other cases are similar. $\square$

Proof of Proposition 4.3. Any $\lambda \in Q^\vee$ gives rise to an infinite biconvex set: we have, for each $\alpha \in \Delta_0^+$,

$$m_\alpha = \begin{cases} \infty & \text{if } \langle \alpha, \lambda \rangle < 0, \\ -\infty & \text{if } \langle \alpha, \lambda \rangle > 0, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear from this that $\mathrm{Inv}(t_\lambda^\infty)$ depends exactly on the face of the braid arrangement that λ lies in. Now let I be any infinite biconvex set, and let $\preceq$ be the pre-order constructed above. Let F be the face of the braid arrangement corresponding to $\preceq$. One checks from the definitions that $\mathrm{Inv}(t_\lambda^\infty)$ and I are in the same block, where $\lambda \in Q^\vee \cap F$. $\square$

Example 4.1. Let us take the face $F = \{(x_1, x_2, x_3, x_4, x_5) \mid x_2 = x_4 < x_1 = x_5 < x_3\}$ as in the example above. The corresponding block is determined by the conditions

$$m_{\alpha_{1,2}} = m_{\alpha_{1,4}} = m_{\alpha_{3,4}} = m_{\alpha_{3,5}} = -\infty,$$

$$m_{\alpha_{1,3}} = m_{\alpha_{2,3}} = m_{\alpha_{2,5}} = m_{\alpha_{4,5}} = \infty,$$

$m_{\alpha_{2,4}}$ and $m_{\alpha_{1,5}}$ are finite. One choice of $\lambda \in Q^\vee \cap F$ is $\lambda = (0, -1, 2, -1, 0)$, the window notation for the corresponding translation is $t_\lambda = [1, -3, 13, -1, 5]$ and one possible reduced factorization is

$$t_\lambda = s_2 s_4 s_3 s_1 s_0 s_4 s_3 s_2 s_1 s_0 s_2 s_1 s_4 s_3.$$

4.4. **Braid limits.** Let $\mathbf{j}$ and $\mathbf{i}$ be infinite words in $\mathbb{Z}/n\mathbb{Z}$. We shall say that $\mathbf{j}$ is a *braid limit* of $\mathbf{i}$ if it can be obtained from $\mathbf{i}$ by a possibly infinite sequence of braid moves. More precisely, we require that one has $\mathbf{i} = \mathbf{j}_0, \mathbf{j}_1, \mathbf{j}_2, \ldots$ such that $\lim_{k\to\infty} \mathbf{j}_k = \mathbf{j}$ and each $\mathbf{j}_k$ differs from $\mathbf{j}_{k+1}$ by finitely many braid-moves. Here, the limit $\lim_{k\to\infty} \mathbf{j}_k = \mathbf{j}$ of words is taken coordinate-wise: $j_r = \lim_{k\to\infty} (\mathbf{j}_k)_r$. We write $\mathbf{i} \to \mathbf{j}$ to mean there is a braid limit from $\mathbf{i}$ to $\mathbf{j}$.

Lemma 4.5. *Suppose* $\mathbf{i}$ *is an infinite reduced word, and* $\mathbf{i} \to \mathbf{j}$*. Then* $\mathbf{j}$ *is also an infinite reduced word.*

The converse of Lemma 4.5 is false. The following example illustrates this, and also the fact that $\mathbf{i} \to \mathbf{j}$ does not imply $\mathbf{j} \to \mathbf{i}$.

Example 4.2. Let $n = 3$, and $\mathbf{i} = 1(012)^\infty = 1012012012\cdots$, which one can check is reduced. Then $\mathbf{i} \to \mathbf{j} = (012)^\infty = 012012012\cdots$:

$$\begin{aligned} &\underline{101}2012012\cdots \\ \sim\ &01\underline{020}12012\cdots \\ \sim\ &0120\underline{212}012\cdots \\ \sim\ &012012\underline{101}2\cdots \\ \sim\ &\cdots \end{aligned}$$

However, there is no braid limit from $\mathbf{j}$ to $\mathbf{i}$ since no braid moves can be performed on $\mathbf{j}$ at all. The same calculation also shows that $11012012012\cdots \to \mathbf{i}$.

Two infinite reduced words $\mathbf{i}$ and $\mathbf{j}$ are called *braid-equivalent* if there are braid limits $\mathbf{i} \to \mathbf{j}$ and $\mathbf{j} \to \mathbf{i}$. Indeed, braid limits define a preorder on the set of all infinite reduced words, and the equivalence classes of this preorder are exactly braid-equivalent infinite reduced words.

Lemma 4.6. *Let $\mathbf{i}$ and $\mathbf{j}$ be infinite reduced words. We have $\mathrm{Inv}(\mathbf{j}) \subset \mathrm{Inv}(\mathbf{i})$ if and only if there is a braid limit from $\mathbf{i}$ to $\mathbf{j}$.*

Proof. Assume there is a braid limit $\mathbf{i} = \mathbf{j}_0, \mathbf{j}_1, \mathbf{j}_2, \ldots$ from $\mathbf{i}$ to $\mathbf{j}$. For every initial part $w_{\mathbf{j}}^{(m)}$ there is a large enough k such that $(\mathbf{j}_k)_r = \mathbf{j}_r$ for $r = 1, \ldots, m$. Since in passing from $\mathbf{i} = \mathbf{j}_0$ to $\mathbf{j}_k$ only finitely many braid moves happened, $w_{\mathbf{j}}^{(m)}$ is initial for some $w_{\mathbf{i}}^{(M)}$, and thus $\mathrm{Inv}(w_{\mathbf{j}}^{(m)}) \subset \mathrm{Inv}(w_{\mathbf{i}}^{(M)})$. Since such M can be found for any m, we conclude that $\mathrm{Inv}(\mathbf{j}) \subset \mathrm{Inv}(\mathbf{i})$.

Assume now $\mathrm{Inv}(\mathbf{j}) \subset \mathrm{Inv}(\mathbf{i})$. Since s_{j_1} is initial in $\mathbf{j}$, the corresponding simple root $\alpha_{j_1} \in \mathrm{Inv}(\mathbf{j})$ and thus $\alpha_{j_1} \in \mathrm{Inv}(\mathbf{i})$. Thus for sufficiently large m, the simple root $\alpha_{j_1} \in \mathrm{Inv}(w_{\mathbf{i}}^{(m)})$. Let $\mathbf{j}_1$ be obtained from $\mathbf{i}$ by applying braid moves to the first m factors to place s_{j_1} in front. Now apply (4) to $\mathbf{j} = (j_1)(j_2 j_3 \cdots)$ and $\mathbf{j}_1 = (j_1)(j'_1 j'_2 \cdots)$ to see that $\mathrm{Inv}(j_2 j_3 \cdots) \subset \mathrm{Inv}(j'_1 j'_2 \cdots)$. Repeating the argument, we construct a braid limit from $\mathbf{i}$ to $\mathbf{j}$. □

Corollary 4.7. *Suppose $\mathbf{i}$ and $\mathbf{j}$ are two infinite reduced words. Then $\mathrm{Inv}(\mathbf{i}) = \mathrm{Inv}(\mathbf{j})$ if and only if they are braid-equivalent.*

4.5. **Exchange lemma for infinite reduced words.** The following result follows immediately from the usual exchange lemma [Hum].

Lemma 4.8. *Let $\mathbf{i}$ be an infinite reduced word and $j \in \mathbb{Z}/n\mathbb{Z}$. Then either*

(1) *$j\mathbf{i}$ is an infinite reduced word, or*
(2) *there is a unique index k such that $\mathbf{i}' = i_1 i_2 \cdots i_{k-1} i_{k+1} \cdots$ is reduced and such that $s_j w_{\mathbf{i}}^{(k)} = w_{\mathbf{i}}^{(k-1)}$.*

For example, let $n = 3$ and let $\mathbf{i} = (012)^2 1 (012)^\infty$. Then $s_1 w_{\mathbf{i}} = (s_0 s_1 s_2)^\infty$, so that $k = 7$, while $s_2 w_{\mathbf{i}}$ is reduced.

Let $\mathbf{i}$ and $\mathbf{j}$ be two infinite reduced words. We say that $\mathbf{j}$ is obtained from $\mathbf{i}$ by *infinite exchange* if Case (2) of Lemma 4.8 always occurs when we place j_1 in front of $\mathbf{i}$, then place j_2 in front of the resulting $\mathbf{i}'$, and so on. For example, with $\mathbf{i}$ and $\mathbf{j}$ as in Example 4.2, $\mathbf{j}$ is obtained from $\mathbf{i}$ by infinite exchange: $j_1 = 0$ is exchanged for the second 1 in $\mathbf{i}$, then

$j_2 = 1$ is exchanged for the first 1 in $\mathbf{i}$, then $j_3 = 2$ is exchanged for the second 0 in $\mathbf{i}$, and so on. It is straightforward to see that

Proposition 4.9. *There is a braid limit $\mathbf{i} \to \mathbf{j}$ if and only if $\mathbf{j}$ can be obtained from $\mathbf{i}$ by infinite exchange.*

Remark 4.3. If $\mathbf{j}$ is obtained from $\mathbf{i}$ by infinite exchange then every letter of $\mathbf{i}$ is eventually "exchanged". (However, the analogous statement fails for arbitrary Coxeter groups.)

4.6. **Limit weak order.** We call a braid-equivalence class $[\mathbf{i}]$ of infinite reduced words a *limit element* of $\tilde{W}$. We let $\tilde{\mathcal{W}}$ denote the set of limit elements of $\tilde{W}$. We define a partial order, called the *limit weak order*, on $\tilde{\mathcal{W}}$ by

$$[\mathbf{i}] \le [\mathbf{j}] \Leftrightarrow \mathrm{Inv}(\mathbf{i}) \subset \mathrm{Inv}(\mathbf{j}).$$

Equivalently, by Lemma 4.6, $[\mathbf{i}] \le [\mathbf{j}]$ if and only if there is a braid limit $\mathbf{j} \to \mathbf{i}$. This partial order does not appear to have been studied before. It is clear that one also obtains a partial order on $\tilde{W} \cup \tilde{\mathcal{W}}$.

Theorem 4.10. *The partial order $(\tilde{W} \cup \tilde{\mathcal{W}}, \le)$ is a meet semi-lattice.*

Proof. It is known ([BB, Theorem 3.2.1]) that $\tilde{W}$ is a meet semi-lattice. Let us take infinite reduced words $\mathbf{i}$ and $\mathbf{j}$ (the other case of $w \in \tilde{W}$ and $\mathbf{i} \in \tilde{\mathcal{W}}$ is similar). Let w_k be the initial part of length k of $\mathbf{i}$, and let u_k be the initial part of length k of $\mathbf{j}$. Then since weak order on the affine symmetric group is a meet-semilattice, we can define $v_k = w_k \wedge u_k$, where $\wedge$ denotes the meet operation. Then $v_k \le v_{k+1}$, since $v_k \le w_k \le w_{k+1}$ and $v_k \le u_k \le u_{k+1}$. If the sequence $v_1 \le v_2 \le \cdots$ stabilizes then we obtain an element w of $\tilde{W}$. Otherwise, we obtain an element $[\mathbf{k}]$ of $\tilde{\mathcal{W}}$. It is clear that w or $[\mathbf{k}]$ is indeed the maximal lower bound of $[\mathbf{i}]$ and $[\mathbf{j}]$. $\square$

We say that $\mathbf{i}$ and $\mathbf{j}$, or $[\mathbf{i}]$ and $[\mathbf{j}]$, are *in the same block* if $\mathrm{Inv}(\mathbf{i})$ and $\mathrm{Inv}(\mathbf{j})$ are. The partial order $(\tilde{\mathcal{W}}, \le)$ descends to blocks: we have $B \le B'$ if one can find $[\mathbf{i}] \in B$ and $[\mathbf{j}] \in B'$ such that $[\mathbf{i}] \le [\mathbf{j}]$. It is convenient to also consider $\tilde{W}$ as a block by itself. The following strengthening of Proposition 4.3 is immediate.

Theorem 4.11. *The map of Proposition 4.3 gives a poset isomorphism of the partial order of blocks of $\tilde{W} \cup \tilde{\mathcal{W}}$ and the inclusion order of the faces of the braid arrangement.*

Note that the maximal blocks of $\tilde{\mathcal{W}}$ consist of single elements, which are the maximal elements of $\tilde{\mathcal{W}}$.

If we label the faces of the braid arrangement by set compositions, then the inclusion order on faces is the refinement order on set compositions: $\Gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k) \preceq \Gamma' = (\gamma'_1, \gamma'_2, \ldots, \gamma'_r)$ if and only if $\gamma_1 = \cup_{i=1}^{r_1} \gamma'_i$, $\gamma_2 = \cup_{i=r_1+1}^{r_1+r_2} \gamma'_i$, and so on.

The infinite translation elements t^∞ are exactly the minimal elements in a block. More generally, if $w \in \tilde{W}$ is so that w^∞ is reduced, then $[w^\infty]$ is minimal in its block. To see this, write $w = vt$ where $v \in W$. Then for some m we have $v^m = 1$, so that w^m will be a translation element.

Theorem 4.12 (cf. [CP, Proposition 3.9] [Ito])**.** *Suppose a block B of $\tilde{\mathcal{W}}$ corresponds to a set composition with sets of sizes $a_1, a_2, \ldots, a_k$, where $\sum_i a_i = n$. Then the restricted partial order $(B, \le)$ is isomorphic to the product $\tilde{S}_{a_1} \times \tilde{S}_{a_2} \times \cdots \times \tilde{S}_{a_k}$ of weak orders of (smaller) affine symmetric groups. In particular, limit weak order is graded when restricted to a block.*

Proof. Fixing the block B fixes some infinite or negative infinite values of certain m_α-s. To check if an assignment of specific finite values to the remaining m_α-s is biconvex, it suffices to check only the triples $\alpha+\beta=\gamma$ such that all three values m_α, m_β, and m_γ are finite. Such finite values correspond to the equivalence classes of the total pre-order $\preceq_B$ on $[n]$ associated to B, or equivalently, to the parts of the set-composition Γ. For each part $\gamma \subset [n]$ of Γ one has to choose finite m_α-s corresponding to an element of $\tilde{S}_{|\gamma|}$. $\square$

Example 4.4. Let $\Gamma = (\{2,4,5\},\{1,3\})$ and let B be the corresponding block. Then an element of B is uniquely determined by the (finite) values of $m_{\alpha_{2,4}}$, $m_{\alpha_{2,5}}$, $m_{\alpha_{4,5}}$ and $m_{\alpha_{1,3}}$. The first three values determine an element of $\tilde{S}_3$, while the last one determines an element of $\tilde{S}_2$. Thus this block is isomorphic to the weak order of $\tilde{S}_3 \times \tilde{S}_2$.

Remark 4.5. Theorem 4.11 can be interpreted in terms of the Tits cone in the geometric realization of $\tilde{W}$. An infinite reduced word can be thought of as an infinite sequence of chambers in the Tits cone, starting from the fundamental chamber. Theorem 4.2(3) (together with Corollary 4.7) can be interpreted as saying that every such sequence is braid-equivalent to a sequence which starts off with a finite sequence of moves (determined by some initial Weyl group element), and then heads straight in some direction (determined by the translation element) for the remaining infinite sequence of moves.

One way to pick such a direction is to pick a point on the boundary of the Tits cone, which in this case is simply a hyperplane. The line joining an interior point of a chamber to a non-zero point in the boundary passes through infinitely-many chambers, and gives the trailing infinite sequence of moves. The intersection of the hyperplane arrangement with the boundary of the Tits cone is simply the (finite) braid arrangement (which in some contexts is called the spherical building at infinity). This gives a geometric interpretation of the classification of Theorem 4.11.

4.7. **Explicit reduced words.** Let B be a block corresponding to a set composition $\Gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)$. We now explain how to write down an explicit infinite reduced word for the minimal element of B. Let λ be a point in the open face F corresponding to Γ. Thus $\lambda_i = \lambda_j$ if $i, j \in \gamma_r$ for some r, and $\lambda_i < \lambda_j$ if $i \in \gamma_r$, $j \in \gamma_s$ for $r < s$. For example, if $\Gamma = (\{2,4\},\{1,5\},\{3\})$, we may pick $\lambda = (2,1,3,1,2)$. Here we drop the convention that $\sum_i \lambda_i = 0$. We may act on λ with simple generators s_i (acting on positions), where s_0 acts by swapping λ_1 and λ_n.

For simplicity, let us suppose that we have chosen (the unique) λ such that

$$\{\lambda_1, \lambda_2, \ldots, \lambda_n\} = \{1, 2, \ldots, k\},$$

as in the above example. Let us start with λ, act with the s_i, and suppose after a sequence of p moves $s_{i_1}, s_{i_2}, \ldots, s_{i_p}$ we obtain λ again, while adhering to the following conditions:

(A) Acting with s_i creates a descent at each step. In other words, we may act with s_i on λ if $\lambda_i < \lambda_{i+1}$ (indices taken modulo n).

(B) For each $r \in \{1, 2, \ldots, k-1\}$, at some point we swap r with $r+1$.

Proposition 4.13. *Let $w = s_{i_1} s_{i_2} \cdots s_{i_p}$. Then w^∞ is reduced and $\mathrm{Inv}(w^\infty)$ is the minimal element of the block B.*

Proof. To see that w is reduced, decorate the 1's inside λ as $1_1, 1_2, \ldots, 1_r$ from left to right, and similarly for the 2's. We may thus think of λ as an affine permutation under the ordering $1_1 < 1_2 < \cdots < 1_r < 2_1 < 2_2 < \cdots$, with w acting on the right. If

a letter moves from λ_1 to λ_n (resp. λ_n to λ_1), we increase its "winding number" by 1 (resp. decrease by 1). One can check that Condition (A) translates to the fact that the length of the affine permutation is always increasing: whenever we swap a letter a with b where $a < b$, it is always the case that the letter b has a greater winding number than the letter a. It follows that w, and similarly also w^∞ is reduced.

By the comment before Theorem 4.12 it follows that $\mathrm{Inv}(w^\infty) = \mathrm{Inv}(t^\infty)$ for some translation t. If we imagine the letters in λ repeated indefinitely in both directions, we obtain a "sea" of 1's, 2's, 3's, and so on. Condition (B) says that as we act with w, the 2's travel to the left with respect to the 1's, and the 3's travel to the left with respect to the 2's, and so on. This implies that t is in the same face of the braid arrangement as λ. □

Example 4.6. For $\lambda = (2, 1, 3, 1, 2)$ as above one possible choice of w is

$$w = s_2 s_1 s_4 s_0 s_1 s_4 s_3.$$

One can calculate that $w^2 = t_{(0,-1,2,-1,0)}$, noting that $(0, -1, 2, -1, 0)$ is in the same open face of the braid arrangement as λ. The resulting action on λ is

$$(2, 1, 3, 1, 2) \to (2, 3, 1, 1, 2) \to (3, 2, 1, 1, 2) \to (3, 2, 1, 2, 1) \to$$

$$(1, 2, 1, 2, 3) \to (2, 1, 1, 2, 3) \to (2, 1, 1, 3, 2) \to (2, 1, 3, 1, 2),$$

and one easily checks that both conditions (A) and (B) are satisfied.

Example 4.7. For $n = 3$ the face complex of the braid arrangement is dual to the face complex of a hexagon, its edges and vertices correspond to the 12 blocks in $\tilde{\mathcal{W}}$. If we label vertices and edges of the hexagon by the corresponding set compositions, reading them in the circular order would produce the following list: $(\{1\}, \{2\}, \{3\})$, $(\{1, 2\}, \{3\})$, $(\{2\}, \{1\}, \{3\})$, $(\{2\}, \{1, 3\})$, $(\{2\}, \{3\}, \{1\})$, $(\{2, 3\}, \{1\})$, $(\{3\}, \{2\}, \{1\})$, $(\{3\}, \{1, 2\})$, $(\{3\}, \{1\}, \{2\})$, $(\{1, 3\}, \{2\})$, $(\{1\}, \{3\}, \{2\})$, $(\{1\}, \{2, 3\})$. A list of corresponding possible choices of w-s for each of the blocks would be $s_1s_2s_1s_0$, $s_2s_1s_0$, $s_2s_1s_0s_1$, $s_2s_0s_1$, $s_2s_0s_2s_1$, $s_0s_2s_1$, $s_0s_1s_2s_1$, $s_0s_1s_2$, $s_1s_0s_1s_2$, $s_1s_0s_2$, $s_1s_0s_2s_0$, $s_1s_2s_0$.

4.8. **Infinite Coxeter elements.** Recall that a Coxeter element $c \in \tilde{W}$ is an element with a reduced word which uses each $i \in \mathbb{Z}/n\mathbb{Z}$ exactly once. It is a standard fact that Coxeter elements of $\tilde{W}$ are in bijection with acyclic orientations of the Dynkin diagram of $\tilde{W}$, which is a n-cycle labeled by $\mathbb{Z}/n\mathbb{Z}$: the simple generator s_i occurs to the left of s_{i+1} in c if and only if the edge $(i, i+1)$ points from $i+1$ to i. From any such acyclic orientation O, we obtain a set composition Γ_O of $[n]$ with two parts:

$$\Gamma_O = (\{i \mid i-1 \to i \text{ in } O\}, \{i \mid i \to i-1 \text{ in } O\}).$$

Proposition 4.14. *Let c, O, Γ_O correspond under the above bijections. Then the affine permutation $w \in \tilde{W}$ of Proposition 4.13 can be chosen to be c.*

Proof. In the case of a two-part set composition, the vector λ consists of 1's and 2's only. The action of c moves each 2 left to the position of the next 2. □

As a consequence we obtain the following result.

Corollary 4.15. *An infinite Coxeter element c^∞ is reduced.*

Corollary 4.15 was proved by Kleiner and Pelley [KP] in the much more general Kac-Moody setting (see also [Spe]).

We have thus given explicit bijections between the following sets: Coxeter elements of $\tilde{W}$, acyclic orientations of a n-cycle, set compositions of $[n]$ with two parts, total pre-orders on $[n]$ with two equivalence classes, and edges of the braid arrangement.

Example 4.8. The Coxeter element $s_2s_4s_0s_1s_3 \in \tilde{S}_5$ corresponds to the pentagon orientation $1 \longrightarrow 2 \longleftarrow 3 \longrightarrow 4 \longleftarrow 5 \longleftarrow 1$, to the set composition $\Gamma = (\{2,4\},\{1,3,5\})$, to the total pre-order $\{2,4\} \prec \{1,3,5\}$, to the edge $F = \{(x_1,x_2,x_3,x_4,x_5) \mid x_2 = x_4 < x_1 = x_3 = x_5\}$.

An infinite reduced word $\mathbf{i}$, or its equivalence class $[\mathbf{i}]$, is *fully commutative* if no (3-term) braid moves $i(i+1)i \to (i+1)i(i+1)$ can be applied to any $\mathbf{j} \in [\mathbf{i}]$.

Lemma 4.16. *Suppose $\mathbf{i} \to \mathbf{j}$ is a braid limit of infinite reduced words where only commutation moves $i\,j \sim j\,i$ where $|i-j| > 1$ are used. Then $\mathbf{i}$ and $\mathbf{j}$ are braid equivalent.*

Proof. Let $i \in \mathbb{Z}/n\mathbb{Z}$. Between every two occurrences of i in $\mathbf{i}$ one has the reduced word of a (rotation of a) usual finite permutation. It follows any two consecutive occurrences of i cannot be too far apart. Thus in particular, any particular letter in $\mathbf{j}$ only traveled a finite distance from its original position in $\mathbf{i}$. Using this and the definition of braid limit, one can construct a braid limit $\mathbf{j} \to \mathbf{i}$. □

Theorem 4.17. *Let $\mathbf{i}$ be an infinite reduced word. The following are equivalent:*

(1) $[\mathbf{i}] = [c^\infty]$, *where c is a Coxeter element,*
(2) $[\mathbf{i}]$ *is fully commutative,*
(3) $[\mathbf{i}]$ *is a minimal element of $(\tilde{\mathcal{W}}, \leq)$.*

Proof. We have already established the equivalence of (1) and (3). Since in c^∞ there is an occurrence of s_{i-1} and s_{i+1} between any two consecutive occurrences of s_i, it is clear that no braid move can possibly be applied and thus c^∞ is fully commutative, giving (1) implies (2). On the other hand, suppose $\mathbf{i}$ is fully commutative, and we have a braid limit $\mathbf{i} \to \mathbf{j}$. Then only commutation moves occurred in the braid limit $\mathbf{i} \to \mathbf{j}$, so by Lemma 4.16, we have $[\mathbf{j}] = [\mathbf{i}]$. This shows that (2) implies (3). □

5. Ω

5.1. **Infinite products of Chevalley generators.** Let $\mathbf{i} = i_1 i_2 \cdots$ be an infinite reduced word (or simply an infinite word), and $\mathbf{a} = (a_1, a_2, \ldots) \in \mathbb{R}^\infty_{>0}$. By [LPI, Lemma 7.1], the limit

$$e_{\mathbf{i}}(\mathbf{a}) := e_{i_1}(a_1) e_{i_2}(a_2) \cdots = \lim_{k \to \infty} e_{i_1}(a_1) \cdots e_{i_k}(a_k)$$

converges if and only if $\sum_i a_i < \infty$. We let $\ell^1_{>0} \subset \mathbb{R}^\infty_{>0}$ denote the set of infinite sequences of positive real numbers with finite sum. We may consider $e_{\mathbf{i}}$ as a map $\ell^1_{>0} \to U_{\geq 0}$. We let $E_{\mathbf{i}} := \mathrm{im}(e_{\mathbf{i}}) \subset U_{\geq 0}$ denote the image of $e_{\mathbf{i}}$, as in the finite case.

Example 5.1. Take $n = 3$, $0 < a < 1$ and consider the following element of $E_{(012)^\infty}$:

$$X = e_0(1)e_1(1)e_2(1)e_0(a)e_1(a)e_2(a) \ldots e_0(a^k)e_1(a^k)e_2(a^k) \ldots = \prod_{k \geq 0} (e_0(a^k)e_1(a^k)e_2(a^k)).$$

Denote $\eta(i,j) = \sum_{i<r<j}(j-r)$ as r assumes all values in the range that are divisible by 3. For example, $\eta(2,6) = 6-3 = 3$. One can compute that

$$x_{i,j} = a^{\eta(i,j)} \prod_{t=1}^{j-i} (1-a^t)^{-1}.$$

Using this formula, one computes: $X =$

$$\left(\begin{array}{c|ccc|ccc|c}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\hline
\cdots & 1 & \frac{1}{1-a} & \frac{1}{(1-a)(1-a^2)} & \frac{a}{(1-a)(1-a^2)(1-a^3)} & \frac{a^2}{(1-a)(1-a^2)(1-a^3)(1-a^4)} & \frac{a^3}{(1-a)(1-a^2)(1-a^3)(1-a^4)(1-a^5)} & \cdots \\
\cdots & 0 & 1 & \frac{1}{1-a} & \frac{a}{(1-a)(1-a^2)} & \frac{a^2}{(1-a)(1-a^2)(1-a^3)} & \frac{a^3}{(1-a)(1-a^2)(1-a^3)(1-a^4)} & \cdots \\
\cdots & 0 & 0 & 1 & \frac{1}{1-a} & \frac{1}{(1-a)(1-a^2)} & \frac{1}{(1-a)(1-a^2)(1-a^3)} & \cdots \\
\hline
\cdots & 0 & 0 & 0 & 1 & \frac{1}{1-a} & \frac{1}{(1-a)(1-a^2)} & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 1 & \frac{1}{1-a} & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\
\hline
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right)$$

By analogy with Lemma 3.1 it can be shown that $E_{\mathbf{i}} = E_{\mathbf{j}}$ if $[\mathbf{i}] = [\mathbf{j}]$, cf. Corollary 5.6. Other properties that hold in finite case do not extend however. For example, sets $E_{\mathbf{i}}$ and $E_{\mathbf{j}}$ may have non-empty intersection even if $[\mathbf{i}] \neq [\mathbf{j}]$ (Corollary 5.7), and the map $\mathbf{a} \mapsto e_{\mathbf{i}}(\mathbf{a})$ is not always injective (Proposition 6.2).

We define

$$\Omega := \bigcup_{\mathbf{i}} E_{\mathbf{i}}$$

where the union is over all infinite reduced words.

Lemma 5.1. *Every $X \in \Omega$ is doubly-entire and totally positive.*

Proof. That X is doubly-entire follows from [LPI, Lemma 7.2] applied to X and X^{-c}. Now suppose $X \in \Omega$ is not totally positive. Then by [LPI, Lemma 5.8 and Lemma 5.10], $\epsilon_i(X) > 0$ for all i, so X cannot be entire. (See Section 6.2 for the definition of $\epsilon_i(X)$.) $\square$

The following result shows that we do not lose anything by only considering reduced words. The proof will be delayed until Section 7.4.

Proposition 5.2. *We have*

$$\Omega \cup U_{\geq 0}^{\mathrm{pol}} = \bigcup_{\mathbf{i}} E_{\mathbf{i}}$$

where the union is taken over all (not necessarily reduced) infinite or finite words.

5.2. **Braid limits in total nonnegativity.** Suppose $\mathbf{i} \to \mathbf{j}$ is a braid limit of infinite reduced words. Applying (2) and (3) possibly an infinite number of times, we obtain a map $R_{\mathbf{i}}^{\mathbf{j}} : \ell^1_{>0} \to \ell^1_{>0}$. This map is well-defined because by the definition of braid limit, any coordinate of $\mathbf{a}' = R_{\mathbf{i}}^{\mathbf{j}}(\mathbf{a})$ will eventually stabilize; in addition, the moves (2) and (3) preserve the sum of parameters, so the image lies in $\ell^1_{>0}$.

Proposition 5.3. *Let $\mathbf{i}$ and $\mathbf{j}$ be infinite reduced words. The map $R_{\mathbf{i}}^{\mathbf{j}}$ does not depend on the braid limit $\mathbf{i} \to \mathbf{j}$ chosen.*

Proof. Suppose we are given two braid limits $\mathbf{i} \to_1 \mathbf{j}$ and $\mathbf{i} \to_2 \mathbf{j}$. Let $\mathbf{i} = \mathbf{j}_0, \mathbf{j}_1, \mathbf{j}_2, \ldots$ be the sequence of infinite reduced words for $\mathbf{i} \to_1 \mathbf{j}$, and let $\mathbf{i} = \mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2, \ldots$ be the sequence for $\mathbf{i} \to_2 \mathbf{j}$.

Let $\mathbf{c} \in \ell^1_{>0}$ and write $\mathbf{a} = (R_{\mathbf{i}}^{\mathbf{j}})_1(\mathbf{c})$, and $\mathbf{b} = (R_{\mathbf{i}}^{\mathbf{j}})_2(\mathbf{c})$. For each $r > 0$, we shall show that $(a_1, a_2, \ldots, a_r) = (b_1, b_2, \ldots, b_r)$. By the definition of braid limit, we can pick s sufficiently large such that the first r letters in $\mathbf{j}_s$, and in $\mathbf{k}_s$, are both equal to the first r letters in $\mathbf{j}$. Now pick m sufficiently large so that all the braid moves involved in going from $\mathbf{i}$ to $\mathbf{j}_s$, and from $\mathbf{i}$ to $\mathbf{k}_s$ occurs in the first m letters. Then $w_{\mathbf{j}_s}^{(m)} = w_{\mathbf{k}_s}^{(m)}$, so that $(\mathbf{j}_s)_1(\mathbf{j}_s)_2 \cdots (\mathbf{j}_s)_m$ can be changed to $(\mathbf{k}_s)_1(\mathbf{k}_s)_2 \cdots (\mathbf{k}_2)_m$ via finitely many braid moves, not involving the first r letters. Using Lemma 3.5, this shows that $(a_1, a_2, \ldots, a_r) = (b_1, b_2, \ldots, b_r)$. $\square$

Proposition 5.4. *For braid limits* $\mathbf{i} \to \mathbf{j} \to \mathbf{k}$ *we have* $R_{\mathbf{i}}^{\mathbf{k}} = R_{\mathbf{j}}^{\mathbf{k}} \circ R_{\mathbf{i}}^{\mathbf{j}}$.

Proof. A pair of braid limits $\mathbf{i} \to \mathbf{j} \to \mathbf{k}$ gives rise to a braid limit $\mathbf{i} \to \mathbf{k}$, obtained by interspersing the braid moves used in $\mathbf{i} \to \mathbf{j}$, and those used in $\mathbf{j} \to \mathbf{k}$. $\square$

The following result is one of our main theorems. We shall give two proofs of this result, in Sections 7.5 and 8.

Theorem 5.5 (TNN braid limit theorem)**.** *Suppose* $\mathbf{i} \to \mathbf{j}$ *is a braid limit. Then* $e_{\mathbf{i}}(\mathbf{a}) = e_{\mathbf{j}}(R_{\mathbf{i}}^{\mathbf{j}}(\mathbf{a}))$.

Remark 5.2. While Theorem 5.5 is an obvious analogue of the Lemma 3.1 for finite reduced words, it is not true in greater generality: it fails when considered in arbitrary Kac-Moody groups.

Corollary 5.6. *Suppose* $\mathbf{i}$ *and* $\mathbf{j}$ *are braid equivalent infinite reduced words. Then* $E_{\mathbf{i}} = E_{\mathbf{j}}$.

Corollary 5.7. *Suppose* $[\mathbf{i}] \leq [\mathbf{j}]$ *in* $(\tilde{\mathcal{W}}, \leq)$*. Then* $E_{[\mathbf{j}]} \subset E_{[\mathbf{i}]}$.

Corollary 5.8. *We have* $\Omega = \cup_c E_{c^\infty}$*, where the union is over all Coxeter elements* c.

Proof. Follows immediately from Theorem 4.17 and Corollaries 5.6 and 5.7. $\square$

Example 5.3. The union in Corollary 5.8 is not in general disjoint. If $c \neq c'$ it is possible to have $E_{c^\infty} \cap E_{(c')^\infty} \neq \emptyset$. For example, take $n = 3$. Then by Example 4.7 one has $(1012)^\infty \to (012)^\infty$ and $(1012)^\infty \to (102)^\infty$. Thus $E_{(1012)^\infty} \subset E_{(012)^\infty} \cap E_{(102)^\infty}$.

We will present Ω as a disjoint union in Section 7.6.

6. Injectivity

By Theorem 3.4, the maps

$$e_{\mathbf{i}} : \mathbb{R}^{\ell}_{>0} \to E_w$$

are injective for a reduced word $\mathbf{i}$ of $w \in \tilde{W}$. The same is not true for the maps $e_{\mathbf{i}} : \ell^1_{>0} \to E_{\mathbf{i}}$.

Example 6.1. Take $n = 3$ and consider the braid limit $\mathbf{i} = 1(012)^\infty \to (012)^\infty = \mathbf{j}$ described in Example 4.2. Then using Theorem 5.5 we obtain

$$e_{\mathbf{i}}(a_1, a_2, \ldots) = e_1(a) e_{\mathbf{i}}(a'_1, a_2, \ldots) = e_1(a) e_{\mathbf{j}}(\mathbb{R}_{\mathbf{i}}^{\mathbf{j}}(a'_1, a_2, \ldots)) = e_{\mathbf{i}}(a, \mathbb{R}_{\mathbf{i}}^{\mathbf{j}}(a'_1, a_2, \ldots))$$

where $0 < a < a_1$ is arbitrary and $a_1 = a + a'_1$. We generalize this in Proposition 6.2 below.

Similarly, $R_{\mathbf{i}}^{\mathbf{j}}$ is a bijection when $\mathbf{i}$ and $\mathbf{j}$ are finite reduced words, but is neither injective nor surjective for general infinite reduced words (see Remark 6.5).

6.1. **Injective reduced words, and injective braid limits.** Let $\mathbf{i}$ be an infinite reduced word. Then $\mathbf{i}$ is *injective* if the map $e_{\mathbf{i}}$ is injective. We shall also say that a braid limit $\mathbf{i} \to \mathbf{j}$ is injective if $R_{\mathbf{i}}^{\mathbf{j}}$ is injective.

Proposition 6.1. *Injectivity of infinite reduced words depends only on the braid equivalence class.*

Proof. Let $\mathbf{i}$ and $\mathbf{j}$ be braid equivalent infinite reduced words. Suppose $\mathbf{i}$ is not injective, so that $e_{\mathbf{i}}(\mathbf{a}) = e_{\mathbf{i}}(\mathbf{a}')$ for some $\mathbf{a} \neq \mathbf{a}'$. Then by Theorem 5.5 we have $e_{\mathbf{j}}(R_{\mathbf{i}}^{\mathbf{j}}(\mathbf{a})) = e_{\mathbf{j}}(R_{\mathbf{i}}^{\mathbf{j}}(\mathbf{a}'))$. By Proposition 5.4, we have $R_{\mathbf{j}}^{\mathbf{i}}(R_{\mathbf{i}}^{\mathbf{j}}(\mathbf{a})) = \mathbf{a} \neq \mathbf{a}' = R_{\mathbf{j}}^{\mathbf{i}}(R_{\mathbf{i}}^{\mathbf{j}}(\mathbf{a}'))$. Thus $\mathbf{j}$ is not injective either. □

Proposition 6.2. *Let $\mathbf{i}$ be an infinite reduced word which is not minimal in its block, and let $X \in E_{\mathbf{i}}$. Then $e_{\mathbf{i}}^{-1}(X) \subset \ell^1_{>0}$ is uncountable. In particular, $\mathbf{i}$ is not injective.*

Proof. Let us say that $\mathbf{i}$ has *rank* $\rho(\mathbf{i}) = p$ if $|\mathrm{Inv}(\mathbf{i}) - \mathrm{Inv}(\mathbf{j})| = p$, where $\mathbf{j}$ is the minimal element in the block of $\mathbf{i}$. Using Theorem 4.2, we may write $\mathbf{i} = s_{i_1} s_{i_2} \cdots s_{i_r} t^{\infty}$. For any reduced expression $i\,\mathbf{k}$, we have $\rho(i\,\mathbf{k}) - \rho(\mathbf{k}) \in \{0, 1\}$. It follows that we may write $\mathbf{i} = u\, i\, \mathbf{j}$, where $\mathbf{j} = t_\lambda^\infty$ is minimal in its own block, and $\rho(i\,\mathbf{j}) = 1$.

It suffices to prove the claim for the case that u is trivial, since prepending u would not change the non-injectivity. Now if $\mathbf{i} = i\,\mathbf{j}$ is reduced and rank 1, then neither α_i not $\delta - \alpha_i$ lies in $\mathrm{Inv}(\mathbf{j})$. It follows that $\langle \alpha_i, \lambda \rangle = 0$, or equivalently, $s_i \lambda = \lambda$. (This calculation holds even if $i = 0$, where for example the inner product is calculated by setting $\delta = 0$, giving $\langle -\alpha_{1,n}, \lambda \rangle = 0$.) But then we have $\mathrm{Inv}(\mathbf{i}) = \{\alpha_i\} \cup s_i \cdot \mathrm{Inv}(\mathbf{j}) = \{\alpha_i\} \cup \mathrm{Inv}(\mathbf{j})$ so that $\mathbf{i} \to \mathbf{j}$. We then have

$$e_{\mathbf{i}}(a_1, a_2, \ldots) = e_1(a) e_{\mathbf{i}}(a_1', a_2, \ldots) = e_1(a) e_{\mathbf{j}}(\mathbb{R}_{\mathbf{i}}^{\mathbf{j}}(a_1', a_2, \ldots)) = e_{\mathbf{i}}(a, \mathbb{R}_{\mathbf{i}}^{\mathbf{j}}(a_1', a_2, \ldots))$$

for any $0 < a < a_1 = a + a_1'$. □

Conjecture 6.3. *Suppose $\mathbf{i}$ is an infinite reduced word which is minimal in its block. Then $e_{\mathbf{i}}$ is injective.*

We shall provide evidence for this conjecture below. In particular, for an infinite Coxeter element $\mathbf{i} = c^\infty$, we will find (many) matrices $X \in E_{\mathbf{i}}$ such that $|e_{\mathbf{i}}^{-1}(X)| = 1$. In view of Corollary 5.8, the case of infinite Coxeter elements is especially interesting.

6.2. **ϵ-sequences and ϵ-signature.** Let $X \in U_{\geq 0}$ be infinitely supported. Recall from [LPI] that for $i \in \mathbb{Z}/n\mathbb{Z}$, we define

$$\epsilon_i(X) = \lim_{j \to \infty} \frac{x_{i,j}}{x_{i+1,j}}.$$

Note that this limit is monotonic: $x_{i,j}/x_{i+1,j} \geq x_{i,j+1}/x_{i+1,j+1} \geq \cdots$. Call $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ the ϵ-sequence of X.

Example 6.2. In Example 5.1 it is clear that $\eta(0,j) = \eta(1,j) = \eta(2,j)$. We compute

$$\epsilon_1 = \lim_{j \to \infty} \frac{a^{\eta(1,j)} \prod_{t=1}^{j-1} (1 - a^t)^{-1}}{a^{\eta(2,j)} \prod_{t=1}^{j-2} (1 - a^t)^{-1}} = \lim_{j \to \infty} (1 - a^{j-1})^{-1} = 1.$$

Similarly $\epsilon_0 = 1$. Finally,

$$\epsilon_2 = \lim_{j\to\infty} \frac{a^{\eta(2,j)} \prod_{t=1}^{j-2}(1-a^t)^{-1}}{a^{\eta(3,j)} \prod_{t=1}^{j-3}(1-a^t)^{-1}} = \lim_{j\to\infty} a^{j-3}(1-a^{j-1})^{-1} = 0.$$

Thus the ϵ-sequence of X is $(1, 0, 1)$.

The sequence of $\{0, +\}$'s arising as signs of the ϵ-sequence is called the ϵ-*signature* of X. By [LPI, Lemma 7.7], for $X \in \Omega$, one cannot have $\epsilon_i(X) > 0$ for all $i \in \mathbb{Z}/n\mathbb{Z}$, so the ϵ-signature has at least one 0.

Example 6.3. For $n = 2$, there are two infinite reduced words $\mathbf{i} = 101010\cdots$ and $\mathbf{j} = 010101\cdots$. For $X \in E_{\mathbf{i}}$, one has the ϵ-signature $(+, 0)$, and for $X \in E_{\mathbf{j}}$, one has $(0, +)$. In this case, the decomposition of 5.8 is disjoint. Furthermore, Conjecture 6.3 holds. If $X = e_1(a_1)e_0(a_2)e_1(a_3)\cdots$ then $\epsilon_1(e_1(-a)X) = 0$, and so we must have $a_1 = \epsilon_1(X)$. Proceeding inductively, we see that $e_{\mathbf{i}}$ is injective.

We first establish some basic results about ϵ-signatures.

Lemma 6.4. *Assume $n > 2$. Let $X \in U_{\geq 0}$ and $i \in \mathbb{Z}/n\mathbb{Z}$. Then for $a > 0$,*

(1) *if $k \neq i, i-1$ then $\epsilon_k(e_i(a)X) = \epsilon_k(X)$;*
(2) *$\epsilon_i(e_i(a)X) = \epsilon_i(X) + a > 0$;*
(3) *$\epsilon_{i-1}(e_i(a)X) > 0$ if and only if $\epsilon_{i-1}(X) > 0$ and $\epsilon_i(X) > 0$;*

Proof. Statements (1) and (2) follow easily from the definition. Statement (3) follows from the following: $\lim_{q\to\infty} \frac{x_{p.q}}{x_{p+1,q}} > 0$ and $\lim_{q\to\infty} \frac{x_{p+1.q}}{x_{p+2,q}} > 0$ if and only if $\lim_{q\to\infty} \frac{x_{p.q}}{x_{p+1,q}+ax_{p+2,q}} > 0$ (where all x's are strictly positive and all limits are known to exist). □

Lemma 6.5. *Let $\mathbf{i}$ be an infinite reduced word and $i \in \mathbb{Z}/n\mathbb{Z}$ be such that the first (leftmost) occurrence of i occurs to the left of the first occurrence of $i+1$ in $\mathbf{i}$. Then for all $X \in E_{\mathbf{i}}$, we have $\epsilon_i(X) > 0$.*

Proof. This follows immediately from Lemma 6.4(1,2). □

Now suppose $[t^\infty]$ is a braid equivalence class of infinite reduced words, minimal in its block. Let $\lambda \in \mathbb{Z}^n$ be the vector used in Subsection 4.7. If $\lambda_i < \lambda_{i+1}$, then s_i is the first simple generator for some $\mathbf{i} \in [t^\infty]$, so by Lemma 6.5 we have $\epsilon_i(X) > 0$ for $X \in E_{t^\infty}$. More generally,

Proposition 6.6. *Let $\Gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)$ be the set composition corresponding to $[t^\infty]$, and let $X \in E_{t^\infty}$. Then $\epsilon_i(X) > 0$ for any $(i+1) \notin \gamma_1$.*

Proof. If $\lambda_{i+1} > 1$, then one can perform the algorithm of Subsection 4.7 in such a way that s_i is performed before s_{i+1}. For example, one can always make $\lambda_i = 1$ (without applying s_{i+1}) and then apply s_i. □

6.3. **Infinite Coxeter factorizations.** In fact, for an infinite Coxeter element c^∞, there are matrices $X \in E_{c^\infty}$ such that Lemma 6.5 completely determines the ϵ-signature of X.

Proposition 6.7. *There is $\mathbf{a} \in \ell^1_{>0}$ such that $X = e_{c^\infty}(\mathbf{a}) \in E_{c^\infty}$ satisfies $\epsilon_i(X) > 0$ if and only if s_i precedes s_{i+1} in c.*

Proof. By Lemma 6.5 the "if" direction holds for all $\mathbf{a}$. Fix a reduced decomposition of c and use it to write down a periodic reduced expression $\mathbf{i} = i_1\, i_2\, i_3\, \cdots$ for c^∞. Let us pick $a_k = \delta^{K+k}$, where $0 < \delta < 1$ and $K > 0$ is a fixed constant. It is clear that $\mathbf{a} \in \ell^1_{>0}$. We wish to calculate $\lim_{s\to\infty} x_{i,i+s}/x_{i+1,i+s}$.

Using [LPI, Section 7.2], $x_{i,s}$ can be expressed as the total weight of certain tableaux T with shape a column of length s: the entries of the tableau T are strictly increasing, and the boxes have residues i, $i+1$, $\ldots$ as we read from the top to the bottom. In a box with residue k, we must place an integer b such that $i_b = k$ (in the terminology of [LPI], one would allow any integer, but if $i_b = k$ is not satisfied, then the weight would be 0). If the boxes of T are filled with numbers $b_1, b_2, \ldots, b_s$, then the weight $\mathrm{wt}(T)$ of T is $a_{b_1} a_{b_2} \cdots a_{b_s}$. [LPI, Lemma 7.3] then states that $x_{i,s} = \sum_T \mathrm{wt}(T)$.

Let S be the set of tableaux enumerated by $x_{i,i+s}$ and S' the set of tableaux enumerated by $x_{i+1,i+s}$. We define a map $\phi : S \to S'$ by removing the first box from T and then reducing all entries by n to obtain T'. This map is well-defined as long as $i+1$ precedes i in $\mathbf{i}$. By our choice of $\mathbf{a}$, we have

$$\mathrm{wt}(T) = a_{b_1}\, \delta^{(s-1)n}\, \mathrm{wt}(T'),$$

where b_1 is the entry of the first box of T. Summing over the possible choices of b_1, we obtain

$$x_{i,i+s} < \left(\sum_r a_r\right) \delta^{(s-1)n}\, x_{i+1,i+s} = \frac{\delta^{K+(s-1)n}}{1-\delta}\, x_{i+1,i+s}.$$

It follows that $\epsilon_i(X) = \lim_{s\to\infty} x_{i,i+s}/x_{i+1,i+s} = 0$, as required. □

Example 6.4. The matrix X from Example 5.1 is clearly an example of such matrix for $c = s_0 s_1 s_2$.

There are however choices of $\mathbf{a}$ such that $X = e_{\mathbf{i}}(\mathbf{a})$ does not satisfy Proposition 6.7.

Proposition 6.8. *Let c be a Coxeter element which is not increasing, that is, of the form $c = s_k s_{k+1} \cdots s_{k-1}$. Then for each $i \in \mathbb{Z}/n\mathbb{Z}$, there is some $X \in E_{c^\infty}$ such that $\epsilon_i(X) > 0$.*

Proof. Let $\Gamma = (\gamma_1, \gamma_2)$ be the (two-part) set composition corresponding to c. The non-increasing condition implies that $|\gamma_1| > 1$. Thus there is a set composition $\Gamma' = (\gamma'_1, \gamma'_2, \gamma'_3)$ of $[n]$ refining Γ, satisfying $\gamma'_3 = \gamma_2$ and $(i+1) \notin \gamma'_1$. The claim then follows from Theorem 5.5 and Proposition 6.6. □

We now use Proposition 6.7 to partition E_{c^∞} into two disjoint parts: $E_{c^\infty} = A_{c^\infty} \sqcup B_{c^\infty}$. Here A_{c^∞} contains the set of matrices with ϵ-signature given by by Lemma 6.5, and B_{c^∞} is the rest of the image. By Proposition 6.7, A_{c^∞} is non-empty. By Proposition 6.8, B_{c^∞} is also non-empty whenever c is not increasing.

We can now prove part of Conjecture 6.3 for infinite Coxeter elements.

Proposition 6.9. *The map e_{c^∞} is injective when restricted to $e_{c^\infty}^{-1}(A_{c^\infty}) \subset \ell^1_{>0}$.*

Proof. Chose a reduced expression for c and let s_i be the first generator in this expression. Then for any $X \in A_{c^\infty}$ we have $\epsilon_i(X) > 0$ and $\epsilon_{i-1}(X) = 0$. We know that $X = e_i(a)Y$ for $Y \in E_{c'^\infty}$, where $c' = s_i c s_i$. By Lemma 6.5 we have $\epsilon_{i-1}(Y) > 0$. By Lemma 6.4(3) we must then have $\epsilon_i(Y) = 0$. This means that $a = \epsilon_i(X)$, and thus the factor $e_i(a)$ of X is unique. Furthermore, one has $Y \in A_{c'^\infty}$, and we may proceed inductively to obtain all the parameters of X. □

We will discuss the topic of injectivity further in Section 10.

Remark 6.5. Propositions 6.7 and 6.8 show that $R_{\mathbf{i}}^{\mathbf{j}}$ is not in general surjective. In fact one can find $\mathbf{i} \to \mathbf{j}$ so that $E_{\mathbf{i}} \subsetneq E_{\mathbf{j}}$: as in the proof of Proposition 6.8, one may find $\mathbf{i}$ so that $\mathbf{i} \to \mathbf{j} = c^\infty$ and $E_{\mathbf{i}} \subset B_{c^\infty}$.

Now consider the braid limit $\mathbf{i} = 1(012)^\infty \to (012)^\infty = \mathbf{j}$ of Example 6.1. We claim that $R_{\mathbf{i}}^{\mathbf{j}}$ is not injective: we have $e_{\mathbf{i}}(\mathbf{a}) = e_{\mathbf{i}}(\mathbf{a}')$ for $\mathbf{a} \neq \mathbf{a}'$. But $(012)^\infty$ is injective since $E_{(012)^\infty} = A_{(012)^\infty}$. Thus $R_{\mathbf{i}}^{\mathbf{j}}(\mathbf{a}) = R_{\mathbf{i}}^{\mathbf{j}}(\mathbf{a}')$.

6.4. The case $n = 3$.

Proposition 6.10. *All infinite Coxeter elements are injective for $n = 3$.*

Proof. By Propositions 6.7 and 6.9, this is the case for the increasing Coxeter elements $c = 012, 120, 201$. Let $X \mapsto X^{-c}$ denote the "c-inverse" involution of [LPI], which in Ω acts by

$$e_{i_1}(a_1)e_{i_2}(a_2)\cdots \mapsto \cdots e_{i_2}(a_2)e_{i_1}(a_1).$$

Now consider the limits $\mu_j(X) = \lim_{i\to-\infty} x_{i,j+1}/x_{i,j}$, applied to $\cdots e_{i_2}(a_2)e_{i_1}(a_1)$. One can check that if $X \in E_{\mathbf{i}}$ then $\mu_j(X^{-c}) > 0$ if j precedes $j-1$ in X. The same arguments as for ϵ's now shows that decreasing infinite Coxeter elements are injective. □

7. ASW factorizations

In this section, we construct for each $X \in \Omega$ a distinguished factorization $X = e_{\mathbf{i}}(\mathbf{a})$, decomposing Ω as a disjoint union of subsets which we call ASW-cells.

We will make use of the following well-known fact (see for example [FH, (15.53)]).

Lemma 7.1 (Three-term Plücker relations)**.** *If Δ_I denotes the minor of a matrix X with row set I and initial column set, then the following identities are true for any set K and distinct $i < j < k < l$ not in K:*

1. $\Delta_{K\cup\{i,k\}}\Delta_{K\cup\{j,l\}} = \Delta_{K\cup\{i,j\}}\Delta_{K\cup\{k,l\}} + \Delta_{K\cup\{i,l\}}\Delta_{K\cup\{j,k\}}$;
2. $\Delta_{K\cup\{i,k\}}\Delta_{K\cup\{j\}} = \Delta_{K\cup\{i,j\}}\Delta_{K\cup\{k\}} + \Delta_{K\cup\{i\}}\Delta_{K\cup\{j,k\}}$.

7.1. q-ASW.

In this section, we assume the reader is familiar with the ASW (Aissen-Schoenberg-Whitney) factorization from [LPI, Section 5].

Let $X \in U_{>0}$ and $q \geq 1$ be an integer. We define the matrix $M_q(X)$ as follows: $m_{q,i,i} = 1$,

$$m_{q,i,j} = (-1)^{j-i} \lim_{l\to\infty} \frac{\Delta_{\{i,\ldots,\hat{j},\ldots,i+q\},\{l,\ldots,l+q-1\}}(X)}{\Delta_{\{i+1,\ldots,i+q\},\{l,\ldots,l+q-1\}}(X)}$$

for $0 < j - i \leq q$ and $m_{q,i,j} = 0$ in all other cases. Here $\hat{j}$ denotes omission of the index j. By [LPI, Theorem 10.6] these limits exist and are finite. Note that $M_1(X) = M(-\epsilon_1(X), -\epsilon_2(X), \ldots, -\epsilon_n(X))$.

Example 7.1. Take the matrix of Example 5.1. We have

$$m_{2,1,3} = (-1)^{3-1} \lim_{l\to\infty} \frac{\Delta_{\{1,2\},\{l.l+1\}}(X)}{\Delta_{\{2,3\},\{l.l+1\}}(X)} =$$

$$= \lim_{l\to\infty} \frac{a^{\eta(2,l)+\eta(2,l)}\left(\prod_{t=1}^{l-1}(1-a^t)^{-2} - (1-a^{l-1})^{-1}(1-a^l)^{-1}\prod_{t=1}^{l-2}(1-a^t)^{-2}\right)}{a^{\eta(2,l)+\eta(2,l)}\left(a^{2-l}\prod_{t=1}^{l-2}(1-a^t)^{-2} - a^{3-l}(1-a^{l-2})^{-1}(1-a^{l-1})^{-1}\prod_{t=1}^{l-3}(1-a^t)^{-2}\right)} =$$

$$= \lim_{l\to\infty} a^{l-2}\frac{(1-a^{l-2})^{-2}(1-a^{l-1})^{-2}-(1-a^{l-2})^{-2}(1-a^{l-1})^{-1}(1-a^{l})^{-1}}{(1-a^{l-2})^{-2}-a(1-a^{l-2})^{-1}(1-a^{l-1})^{-1}} =$$

$$= \lim_{l\to\infty}\frac{a^{2l-3}}{(1-a^{l-1})(1-a^{l})} = 0.$$

In this manner one computes

$$M_2(X) = \left(\begin{array}{c|ccc|ccc|c}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \hline
\dots & 1 & -1 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 1 & -1 & 1 & 0 & 0 & \dots \\
\dots & 0 & 0 & 1 & -1-a & a & 0 & \dots \\ \hline
\dots & 0 & 0 & 0 & 1 & -1 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 1 & -1 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ \hline
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right).$$

Lemma 7.2. *Suppose $X \in U_{>0}$ is totally positive. Then $M_q(X)X \in U_{>0}$.*

Proof. Denote $Y = M_q(X)X$. It is enough to show $Y \in U_{\geq 0}$ since $M_q(X)$ is finitely supported, and the product of a finitely supported TNN matrix and a TNN matrix which is not totally positive, is never totally positive (see [LPI, Theorem 5.5 and Theorem 5.7]).

We claim that for any $j_1 < \ldots < j_{k+1}$

$$\Delta_{\{i,\ldots,i+k\},\{j_1,\ldots,j_{k+1}\}}(Y) = \lim_{l\to\infty}\frac{\Delta_{\{i,\ldots,i+k+q\},\{j_1,\ldots,j_{k+1},l,\ldots,l+q-1\}}(X)}{\Delta_{\{i+k+1,\ldots,i+k+q\},\{l,\ldots,l+q-1\}}(X)}.$$

The fact that this limit exists and is finite is part of the claim to become evident later. For a fixed l, let

$$m^{(l)}_{q,i,j} = (-1)^{j-i}\frac{\Delta_{\{i,\ldots,\hat{j},\ldots,i+q\},\{l,\ldots,l+q-1\}}(X)}{\Delta_{\{i+1,\ldots,i+q\},\{l,\ldots,l+q-1\}}(X)}$$

and let $M^{(l)}_q$ be the matrix filled with entries $m^{(l)}_{q,i,j}$ in rows i through $i+k$, and coinciding with identity elsewhere. Here we assume l is large enough so that all needed $m^{(l)}_{q,i,j}$-s are well-defined. Note that $M^{(l)}_q$ is not infinitely periodic. We have by definition

$$\lim_{l\to\infty} m^{(l)}_{q,i,j} = m_{q,i,j}.$$

We claim that the entries of $Y^{(l)} = M^{(l)}_q X$ in rows i through $i+k$ and in columns l through $l+q-1$ are zero. Assume for now that this is known. We further observe that multiplication by $M^{(l)}_q$ does not change the determinant $\Delta_{\{i,\ldots,i+k+q\},\{j_1,\ldots,j_{k+1},l,\ldots,l+q-1\}}(X)$, since $M^{(l)}_q(X)$ is supported only within the first q diagonals. Therefore

$$\Delta_{\{i,\ldots,i+k\},\{j_1,\ldots,j_{k+1}\}}(Y^{(l)}) = \frac{\Delta_{\{i,\ldots,i+k+q\},\{j_1,\ldots,j_{k+1},l,\ldots,l+q-1\}}(X)}{\Delta_{\{i+k+1,\ldots,i+k+q\},\{l,\ldots,l+q-1\}}(X)}.$$

Taking the limit $l \to \infty$ we obtain the needed statement: the limit on the left exists and equals the limit on the right, which is thus finite. Now we observe that every row-solid minor of Y is a limit of a positive value, and thus is nonnegative. By [LPI, Lemma 2.3] this implies that $Y \in U_{\geq 0}$.

It remains to argue that the mentioned entries of $Y^{(l)}$ are zero. We argue that $y_{i,l}^{(l)} = 0$: this follows from the relation

$$x_{i,l}\Delta_{\{i+1,\ldots,i+q\},\{l,\ldots,l+q-1\}}(X) - x_{i+1,l}\Delta_{\{i,i+2\ldots,i+q\},\{l,\ldots,l+q-1\}}(X)+ \\ \ldots \pm x_{i+q,l}\Delta_{\{i,\ldots,i+q-1\},\{l,\ldots,l+q-1\}}(X) = 0$$

obtained by expanding the determinant of the submatrix $X_{\{i,i+1,\ldots,i+q\},\{l,l,l+1,\ldots,l+q-1\}}$ (note that column l is repeated) along the first column. The same argument works for any choice of a row and a column in the specified range. □

Note that $M_1(X)$ is the ASW factorization applied to X, i.e. $(M_1(X))^{-1}$ is exactly the curl factored out from X by ASW. It is then natural to expect $M_q(X)$ to have some maximality property similar to that of ASW factorization, cf. [LPI, Lemma 5.4]. This is made precise by the following lemma.

Lemma 7.3. *Among all matrices M supported on first q diagonals such that $MX \in U_{\geq 0}$ the matrix $M_q(X)$ has minimal (most negative) entries directly above the diagonal.*

Proof. Consider the ratio

$$\frac{\Delta_{\{i,i+2,\ldots,i+q\},\{l,\ldots,l+q-1\}}(X)}{\Delta_{\{i+1,\ldots,i+q\},\{l,\ldots,l+q-1\}}(X)}.$$

When multiplying by M on the left, only the entry $m_{i,i+1}$ will affect this ratio, since the next $q-1$ entries $m_{i,j}$ in that row do not influence either determinant, while beyond that M is zero. By [LPI, Lemma 10.5], the limits defining $M_q(X)$ are monotonic. Thus $m_{q,i,i+1}$ is the minimal value such that in MX the above ratio remains nonnegative for all values of l. □

Lemma 7.4. *The matrix $M_q(X)X$ is equal to the matrix obtained by q iterations of ASW factorization on X.*

Proof. We show that $M_1(M_{q-1}(X)X)M_{q-1}(X) = M_q(X)$, and the result will follow by induction on q.

To simplify the notation we denote $\Delta_I = \Delta_{I,\{l,\ldots,l+|I|-1\}}(X)$. Let

$$a_i^{(l)} = m_{q,i,i+1}^{(l)} - m_{q-1,i,i+1}^{(l)} = -\frac{\Delta_{\{i,i+2,\ldots,i+q\}}}{\Delta_{\{i+1,\ldots,i+q\}}} + \frac{\Delta_{\{i,i+2,\ldots,i+q-1\}}}{\Delta_{\{i+1,\ldots,i+q-1\}}} = -\frac{\Delta_{\{i,\ldots,i+q-1\}}\Delta_{\{i+2,\ldots,i+q\}}}{\Delta_{\{i+1,\ldots,i+q-1\}}\Delta_{\{i+1,\ldots,i+q\}}}$$

which is evidently negative. Let $a_i = \lim_{l\to\infty} a_i^{(l)}$. We claim that $M(a_1,\ldots,a_n)M_{q-1}(X) = M_q(X)$, the entries directly above the diagonal coincide by definition of a_i-s. For the rest of the entries, we perform the following calculation, using Lemma 7.1:

$$\left(-\frac{\Delta_{\{i,i+2,\ldots,i+q\}}}{\Delta_{\{i+1,\ldots,i+q\}}} + \frac{\Delta_{\{i,i+2,\ldots,i+q-1\}}}{\Delta_{\{i+1,\ldots,i+q-1\}}}\right)\frac{\Delta_{\{i+1,\ldots,\hat{j},\ldots,i+q\}}}{\Delta_{\{i+2,\ldots,i+q\}}} - \frac{\Delta_{\{i,\ldots,\hat{j},\ldots,i+q-1\}}}{\Delta_{\{i+1,\ldots,i+q-1\}}} =$$
$$= -\frac{\Delta_{\{i,i+2,\ldots,i+q\}}\Delta_{\{i+1,\ldots,\hat{j},\ldots,i+q\}}}{\Delta_{\{i+1,\ldots,i+q\}}\Delta_{\{i+2,\ldots,i+q\}}} + \frac{\Delta_{\{i,i+2,\ldots,\hat{j},\ldots,i+q\}}}{\Delta_{\{i+2,\ldots,i+q\}}} = -\frac{\Delta_{\{i,\ldots,\hat{j},\ldots,i+q\}}}{\Delta_{\{i+1,\ldots,i+q\}}},$$

which means $a_i^{(l)}m_{q-1,i+1,j}^{(l)} + m_{q-1,i,j}^{(l)} = m_{q,i,j}^{(l)}$, and passing to a limit $a_i m_{q-1,i+1,j} + m_{q-1,i,j} = m_{q,i,j}$ as desired.

Next, we claim that $M(a_1,\ldots,a_n) = M_1(M_{q-1}(X)X)$. Indeed, by Lemma 7.2 and the calculation above we know that the curl $M(a_1,\ldots,a_n)^{-1}$ can be factored out from $M_{q-1}(X)X$ so that the result is totally nonnegative. On the other hand, by Lemma 7.3

we see that each parameter a_i is minimal possible for which such factorization could exist. This means that $M(a_1, \ldots, a_n)^{-1}$ is exactly the result of ASW factorization applied to $M_{q-1}(X)X$. $\square$

Example 7.2. The matrix $M_2(X)$ obtained in Example 7.1 factors as

$$M_2(X) = \left(\begin{array}{c|ccc|ccc|c} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \hline \cdots & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & -1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & -a & 0 & 0 & \cdots \\ \hline \cdots & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & -1 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \hline & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right) \left(\begin{array}{c|ccc|ccc|c} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \hline \cdots & 1 & -1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & -1 & 0 & 0 & \cdots \\ \hline \cdots & 0 & 0 & 0 & 1 & -1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \hline & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

The two factors are exactly the results of the usual ASW factorization applied to X twice.

One way to interpret Lemmata 7.3 and 7.4 is to say that the local maximality of ASW factorization translates to global maximality: the maximal way to factor out a product of q curls is to greedily factor out a maximal curl at each step. We use this to derive the following property of ASW factorization on Ω.

Theorem 7.5. *Let $X \in \Omega$ and let $N_1, N_2, \ldots$ be degenerate curls obtained by repeated application of ASW factorization to X. Then $X = \prod_{i\geq 1} N_i$.*

Proof. First, it is clear that $\prod_{i\geq 1} N_i$ exists and is $\leq X$ entry-wise. It suffices then to show that for any initial part $X^{(k)} = \prod_{j=1}^{k} e_{i_j}(a_j)$ of X we have $\prod_{i\geq 1} N_i \geq X^{(k)}$ entry-wise. In fact, it is enough to check this latter inequality for only the entries directly above the diagonal, since $(\prod_{i\geq 1} N_i)^{-1} X \in U_{\geq 0}$, and a TNN matrix which has 0's directly above the diagonal is the identity matrix.

This however follows from Lemma 7.3: since $X^{(k)} = \prod_{j=1}^{k} e_{i_j}(b_j)$ is a product of k curls, the product $\prod_{i=1}^{k} N_i$ has greater entries just above the diagonal. $\square$

Lemma 7.6. *Let $X \in \Omega$. If the ASW factorization is $X = e_{\mathbf{i}}(\mathbf{a})$, then $\mathbf{i}$ is necessarily an infinite reduced word.*

Proof. This is a special case of Lemma 9.5. $\square$

7.2. **ASW factorization for finitely supported matrices.** Let $X \in U_{\geq 0}$ be finitely supported matrix such that X^c is entire. One can define a finite version of ASW factorization as follows. For a given i, let j be maximal such that $x_{i+1,j} \neq 0$. Define $\epsilon_i(X) = \frac{a_{i,j}}{a_{i+1,j}}$. It is clear that not all the ϵ_i can be simultaneously 0, otherwise by [LPI, Theorem 5.5] X would be a product of non-degenerate whirls, and X^c would not be entire.

Just as for the infinitely supported case, we will call the factorization in the following Proposition *ASW factorization.*

Proposition 7.7. *Let $U_{\geq 0}$ be finitely supported.*

(1) *$M(-\epsilon_1, \ldots, -\epsilon_n)X \in U_{\geq 0}$, or in other words the degenerate curl $N(\epsilon_1, \ldots, \epsilon_n)$ can be factored out from X;*
(2) *for any other curl $N(a_1, \ldots, a_n)$ that can be factored out from X we have $a_i \leq \epsilon_i$.*

Proof. We follow the strategy in the proof of [LPI, Theorem 2.6]. It is clear that the set of non-zero entries in X has some NE corners, which implies that some of ϵ_i-s are zero. We can group all i-s with non-zero ϵ_i-s into sets that share common j in the definition of ϵ_i above. This divides $\mathbb{Z}/n\mathbb{Z}$ into a number of cyclic intervals. It was essentially shown in the proof of [LPI, Theorem 2.6] that the Chevalley generators corresponding to i-s initial (smallest) in those intervals can be factored from X with parameters equal to the corresponding ϵ_i-s. One can then iterate this argument to factor Chevalley generators corresponding to non-initial elements of the intervals. The resulting collection of Chevalley generators factored has product equal to $N(\epsilon_1, \ldots, \epsilon_n)$, as desired. The second statement is clear since if $a_i > \epsilon_i$ for some $i \in \mathbb{Z}/n\mathbb{Z}$, the product $M(-a_1, \ldots, -a_n)X$ would have a negative entry in row i. □

The following result is the finitely supported analogue of Theorem 7.5.

Proposition 7.8. *Assume $X \in U_{\geq 0}$ is a finite product of Chevalley generators. Then repeated application of ASW factorization results in an expression $X = N_1 \ldots N_l$ of X as a product of degenerate curls.*

Proof. If there was a non-trivial remainder in the above ASW factorization, by [LPI, Theorem 5.5] this remainder would be a finite product of non-degenerate whirls. Then X^{-c} would not be entire, which would be a contradiction. □

7.3. **Uniqueness of Ω factors.** From the definition, a matrix X is entire if it is either finitely supported, or $\lim_{j\to\infty} x_{i,j}/x_{i+n,j} = 0$ for each i.

Lemma 7.9. *Let $X, Y \in U_{\geq 0}$ be such that X is infinitely supported and Y is entire. Then for any i we have $\epsilon_i(XY) = \epsilon_i(X)$.*

Proof. By [LPI, Lemmata 5.3 and 5.4], one has $\epsilon_i(XY) \geq \epsilon_i(X)$ since if $N(a_1, a_2, \ldots, a_n)$ can be factored from X then it can also be factored from XY. Thus, it suffices to show that $\epsilon_i(XY) \leq \epsilon_i(X)$.

For convenience of notation let $X = (a_{i,j})_{i,j=-\infty}^{\infty}$, $Y = (b_{i,j})_{i,j=-\infty}^{\infty}$ and $XY = (c_{i,j})_{i,j=\infty}^{\infty}$. Let $\epsilon = \epsilon_i(X)$. For a given $\delta > 0$, let us pick N such that $\frac{a_{i,k}}{a_{i+1,k}} < \epsilon + \delta$ for $k > N$. Choose $C > 0$ such that $C < \frac{a_{i+1,k+n}}{a_{i,k}}\delta$ for $i \leq k \leq N$. Such a C exists since X is infinitely supported. Now pick $j \gg N$ sufficiently large such that $b_{k,j} \leq C b_{k+n,j}$ for $i \leq k \leq N$. This is possible since Y is entire. Then for $k \in [i, N]$,

$$a_{i,k}b_{k,j} \leq a_{i,k}b_{k+n,j}C < \delta a_{i+1,k+n}b_{k+n,j}.$$

We have

$$c_{i,j} = \sum_{k=i}^{j} a_{i,k}b_{k,j} = \sum_{k=i}^{N} a_{i,k}b_{k,j} + \sum_{k=N+1}^{j} a_{i,k}b_{k,j}$$
$$\leq \delta(\sum_{k=i}^{N} a_{i+1,k+n}b_{k+n,j}) + (\epsilon+\delta)(\sum_{k=N+1}^{j} a_{i+1,k}b_{k,j}) < (\epsilon + 2\delta)c_{i+1,j}.$$

This holds for sufficiently large j, and since we can choose δ to be arbitrarily small, we conclude that $\epsilon_i(XY) \leq \epsilon$ as desired. □

Lemma 7.10. *Let $X, Y \in U_{\geq 0}$ be such that X is finitely supported and Y is infinitely supported, and $\epsilon_i(Y) = 0$ for all $i \in \mathbb{Z}/n\mathbb{Z}$. Then for any i we have $\epsilon_i(XY) = \epsilon_i(X)$.*

Proof. As in the proof of the Lemma 7.9, it suffices to show that $\epsilon_i(XY) \leq \epsilon_i(X)$. As before, let $X = (a_{i,j})_{i,j=-\infty}^{\infty}$, $Y = (b_{i,j})_{i,j=-\infty}^{\infty}$ and $XY = (c_{i,j})_{i,j=\infty}^{\infty}$. Let N be such that $a_{i+1,N} > 0$ but $a_{i+1,k} = 0$ for $k > N$. Thus by definition $\epsilon_i(X) = a_{i,N}/a_{i+1,N}$.

Let $\delta > 0$ and choose $C > 0$ such that $a_{i,k}C < a_{i+1,k+1}\delta$ for $i \leq k \leq N-1$. Next, choose j large enough such that $\frac{b_{k,j}}{b_{k+1,j}} < C$ for $i \leq k \leq N-1$. This can be done since Y is infinitely supported and $\epsilon_r(Y) = 0$ for any r. Then we can write

$$c_{i,j} = \sum_{k=i}^{j} a_{i,k}b_{k,j} = \sum_{k=i}^{N} a_{i,k}b_{k,j} = \sum_{k=i}^{N-1} a_{i,k}b_{k,j} + a_{i,N}b_{N,j} < \sum_{k=i}^{N-1} a_{i,k}b_{k+1,j}C + \epsilon a_{i+1,N}b_{N,j}$$

$$< \sum_{k=i}^{N-1} a_{i+1,k+1}b_{k+1,j}\delta + \epsilon a_{i+1,N}b_{N,j} < \delta c_{i+1,j} + \epsilon c_{i+1,j} = (\delta+\epsilon)c_{i+1,j}.$$

This holds for sufficiently large j, and since we can choose δ to be arbitrarily small, we conclude that $\epsilon_i(XY) \leq \epsilon_i(X)$ as desired. □

Define $\mathbb{L}_r$ to be the right limit-semigroup generated by Chevalley generators (see [LPI, Section 8.3]). In other words, $\mathbb{L}_r \subset U_{\geq 0}$ is the smallest subset of $U_{\geq 0}$ which contains Chevalley generators $\{e_i(a) \mid a \geq 0\}$, and is closed under products, and right-infinite products. For example, $\mathbb{L}_r$ contains matrices of the form $X = \prod_{i=1}^{\infty} X^{(i)}$ where each $X^{(i)}$ lies in Ω.

The following result proves that the factorization of [LPI, Theorem 8.8] is unique. Recall the definition of $\mu_i(X)$ from [LPI], or the proof of Proposition 6.10.

Theorem 7.11.

(1) *Let $X \in U_{\geq 0}$ be entire. There is a unique factorization $X = YZ$, where $Y \in \mathbb{L}_r$ and Z satisfies $\epsilon_i(Z) = 0$ for each $i \in \mathbb{Z}/n\mathbb{Z}$.*

(2) *Let $X \in U_{\geq 0}$ be such that X^{-c} is entire. There is a unique factorization $X = Z'Y'$, where $Y' \in \mathbb{L}_l = (\mathbb{L}_r)^{-c}$ and Z' satisfies $\mu_i(Z') = 0$ for each $i \in \mathbb{Z}/n\mathbb{Z}$.*

Proof. We prove (1), as (2) is similar. By Lemmata 7.9 and 7.10, applying (possibly finite) ASW factorization to Y produces the same result as ASW applied to X. By Theorem 7.5 and Proposition 7.8, this allows one to extract the first Ω-factor of Y (or Y itself, if $Y \in U_{\geq 0}^{\mathrm{pol}}$). Since the first factor in Y is determined by X, by transfinite induction we conclude that all factors are. □

Theorem 7.12.

(1) *when a Chevalley generator is factored from an element of Ω giving a totally non-negative matrix, the resulting matrix also lies in Ω;*

(2) *each element of $\mathbb{L}_r$ has a unique factorization into factors which lie in Ω, with possibly one factor which is a finite product of Chevalley generators;*

(3) *if $X \in \Omega$ and $X = YZ$ where $Y \in \Omega$ and $Z \in U_{\geq 0}$, then Z is the identity matrix.*

Proof. (1) Let $X \in \Omega$ and assume $X = e_i(t)X'$. Apply Theorem 7.11 to write $X' = Y'Z'$. Then $X = e_i(t)Y'Z'$ and by uniqueness in Theorem 7.11 we conclude that $X = e_i(t)Y'$ and $Z' = I$ is the identity matrix. Thus $X' = Y' \in \mathbb{L}_r$. If X' is a finite product of Chevalley generators then X is finitely supported, contradicting $X \in \Omega$. Otherwise $X' = X''X'''$ where $X'' \in \Omega$, $X''' \in \mathbb{L}_r$, possibly $X''' = I$. By Theorem 7.5 and Lemma 7.9, ASW factorization applied repeatedly to X and

$e_i(t)X''$ produces the same result, and this result is equal to both X and $e_i(t)X''$. Thus $X''' = I$ and $X' \in \Omega$, as desired.

(2) If X is a finite product of Chevalley generators, the statement is clear. Otherwise assume $X = YZ$ with $X, Z \in \mathbb{L}_r$, $Y \in \Omega$. By Lemma 7.9 applying ASW repeatedly to X and Y produces the same result, which by Theorem 7.5 is equal to Y. Thus the factor Y of X can be recovered uniquely. By transfinite induction we conclude that every Ω-factor in $X \in \mathbb{L}_r$ is unique.

(3) This follows immediately from (2).

□

7.4. **Proof of Proposition 5.2.** It is clear that $\cup_{\mathbf{i}} E_{\mathbf{i}}$ contains $\Omega \cup U_{\geq 0}^{\mathrm{pol}}$. Let $X = e_{\mathbf{i}}(\mathbf{a})$, where $\mathbf{i}$ is possibly not reduced. We may assume that $\mathbf{i}$ is infinite. Let us apply ASW factorization to X, to obtain $X = YZ$, where $Y \in \Omega \cup U_{\geq 0}^{\mathrm{pol}}$ and $Z \in U_{\geq 0}$. If Z is not the identity matrix, then for some $i \in \mathbb{Z}/n\mathbb{Z}$ we have $s = z_{i,i+1} > 0$. Let $s' = x_{i,i+1} = \sum_{i_r = i} a_r$, which we know is greater than or equal to s. We can find some k sufficiently large that $\sum_{i_r = i | r<k} a_r > s' - s$. The matrix $M = e_{i_k}(-a_k)\cdots e_{i_1}(-a_1)$ is supported on the first k diagonals and MX is TNN. Thus by Lemma 7.3 and Lemma 7.4, the matrix $M_k(X)$ has smaller entries on above the diagonal than M. It follows that $y_{i,i+1} \geq s' - s$. But this contradicts $s = z_{i,i+1}$. We conclude that $Z = I$, and so $X \in \Omega \cup U_{\geq 0}^{\mathrm{pol}}$.

7.5. **First proof of Theorem 5.5.** Let $\mathbf{a}' = R_{\mathbf{i}}^{\mathbf{j}}(\mathbf{a})$. It is clear from the definition of braid limit that $e_{j_1}(a'_1)\cdots e_{j_k}(a'_k)$ can be factored out of $e_{\mathbf{i}}(\mathbf{a})$ on the left. Since limits of TNN matrices are TNN ([LPI, Lemma 2.4]), we deduce that $e_{\mathbf{i}}(\mathbf{a}) = e_{\mathbf{j}}(R_{\mathbf{i}}^{\mathbf{j}}(\mathbf{a}))\,Z$ where $Z \in U_{\geq 0}$. By Theorem 7.12(3), Z is the identity matrix.

7.6. **ASW cells.** Assume $X \in U_{\geq 0}$ and let $\sigma(X) = (\sigma_1(X), \ldots, \sigma_n(X))$ be its ϵ-signature, so that $\sigma_i(X) = \mathrm{sign}(\epsilon_i(X)) \in \{0, +\}$.

Lemma 7.13. *Let $X \in U_{\geq 0}$. Let $X = NY$ be a single application of ASW factorization to X, so that $N = N(\epsilon_1(X), \ldots, \epsilon_n(X))$ is a (possibly degenerate) curl. Then*

$$\{i \mid \sigma_i(Y) = +\} \subseteq \{i \mid \sigma_{i+1}(X) = +\}.$$

Proof. First suppose X is infinitely supported. The statement is trivially true if $\sigma(X)$ consists of all $+$'s. If $\sigma_{i+1}(X) = 0$ then factorizing N our of X does not change the $i+1$-st row of X. The i-th row may or may not change depending on $\sigma_i(X)$. However, in either case the ratio of new i-th row to the old, and thus also to the new $i+1$-st row becomes 0 at the limit, that is $\sigma_i(Y) = 0$.

Now suppose X is finitely supported. We look at the north-east boundary of the non-zero entries of X. If $\epsilon_{i+1}(X) = 0$ then the $i+1$-st row is the same in X and Y. Even if in X the last non-zero entries in the i-th and $i+1$-st rows are in the same column, after factoring N out it is not true anymore, and thus $\epsilon_i(Y) = 0$. □

A degenerate curl N is a finite product of Chevalley generators. Define $v(N) \in \tilde{W}$ by requiring that $N \in E_{v(N)}$. We can describe the possible $v \in \tilde{W}$ that result as follows. A word $i_1 i_2 \cdots i_k$ in the alphabet $\mathbb{Z}/n\mathbb{Z}$ is *cyclically increasing* if no letter is repeated, and whenever i and $i+1$ (taken modulo n) are both present, i occurs before $i+1$. An element $v \in \tilde{W}$ is cyclically increasing if some (equivalently, every) reduced word for v is cyclically increasing. Cyclically increasing elements are exactly the ones occurring as $v(N)$ for some

degenerate curl N. (The reversed notion of cyclically decreasing elements is studied in [Lam].) Note that a cyclically increasing permutation v is completely determined by which simple generators s_i occur, and for $v = v(N(a_1, a_2, \ldots, a_n))$ these are exactly the indices i such that $a_i > 0$.

For $v = s_{i_1} s_{i_2} \cdots s_{i_\ell} \in \tilde{W}$ and an integer $k \geq 0$, let us define $v^{(k)} = s_{i_1-k} s_{i_2-k} \cdots s_{i_\ell-k}$, obtained by rotating the indices. Note that $v^{(k)}$ does not depend on the reduced word of v chosen. Define the infinite "falling power" $v^{[\infty]} = \prod_{k \geq 0} v^{(k)}$, considered as a possibly non-reduced infinite word, assuming that a reduced word for v has been fixed.

By Theorem 7.5, applying ASW factorization to an element $X \in \Omega$ repeatedly leads us to a factorization $X = \prod_{j \geq 1} N_j$ into degenerate curls. By Lemma 7.13, we have $\ell(v(N_1)) \geq \ell(v(N_2)) \geq \cdots$, and at some point the lengths must stabilize: there is some minimal l such that $\ell(v(N_{l+k})) = \ell(v(N_l))$ for every $k \geq 0$. By Lemma 7.13 again, we have in fact $v(N_{l+k}) = v(N_l)^{(k)}$. Thus $X \in E_{wv^{[\infty]}}$, where $v = v(N_{l+1})$ and $w = \prod_{j=1}^{l} v(N_j)$.

Whenever a pair $(w, v) \in \tilde{W} \times \tilde{W}$ occurs in the above manner for some $X \in \Omega$, we say w and v are *compatible*, and write $X \in A(w, v)$. Then ASW factorization decomposes Ω into a disjoint union

$$\Omega = \bigsqcup_{(w,v)} A(w, v)$$

over the set of compatible pairs. We call the sets $A(w, v)$ ASW-cells (even though they may have complicated topology). For the rest of the section, our aim is to describe the set of compatible pairs.

We first introduce a version of ASW factorization at the level of affine permutations. We shall require (strong) Bruhat order ([Hum]) on the affine symmetric group in the following, and shall denote it by $w <_s v$.

Proposition 7.14.

(1) *Let $w \in \tilde{W}$. Then there is a cyclically increasing $v \in \tilde{W}$ such that $v \leq w$ (in weak order), and for any other cyclically increasing $v' \leq w$ we have $v' <_s v$. The same result holds for $\mathbf{i} \in \tilde{\mathcal{W}}$. We call the factorization $w = vu$ (resp. $\mathbf{i} = v\mathbf{j}$) the (combinatorial) ASW factorization of w (resp. $\mathbf{i}$).*

(2) *If $w = v_1 \ldots v_k$ is the result of repeated ASW factorization of $w \in \tilde{W}$, then $v_{i+1} \leq_s v_i^{(1)}$, for $i = 1, \ldots, k-1$.*

Proof. We prove (1). Suppose first that $w \in \tilde{W}$. Choose any $X \in E_w$. Let N be the curl factored from X by ASW factorization, and let $v = v(N)$. We claim that v is the required cyclically increasing element. First, by Theorem 3.4(2) we know that $v \leq w$. Suppose v' is another cyclically increasing element satisfying $v' \leq w$, so that v' is not less than v in Bruhat order. Then there must be a simple generator s_i in v' that is not contained in v. Since v contains all s_i-s such that $\epsilon_i(X) > 0$, it has to be the case that $\epsilon_i(X) = 0$. On the other hand, since $v' \leq w$, one can factor out a curl N' from X satisfying $v(N') = v'$. This would imply $\epsilon_i(X) > 0$, a contradiction. In the case of $\mathbf{i} \in \tilde{\mathcal{W}}$, observe that there are only finitely many cyclically increasing elements in $\tilde{W}$. Thus one can choose a sufficiently large initial part w of $\mathbf{i}$ such that for every cyclically increasing v we have $v < \mathbf{i}$ if and only if $v < w$. This reduces the statement to the established case.

We prove (2). Pick a representative $X \in E_w$. We have just seen that the reduced word of the ASW factorization of X is the same as the reduced word of the (combinatorial) ASW factorization of w. The claim now follows from Lemma 7.13. $\square$

Remark 7.3. The factorization of $w \in \tilde{W}$ into maximal cyclically increasing elements in Proposition 7.14 gives the dominant monomial term of an affine Stanley symmetric function [Lam].

Proposition 7.15. *Suppose $w = v_1 \ldots v_k$ is the result of combinatorial ASW factorization of $w \in \tilde{W}$. A pair (w, v) is compatible if and only if:*

(1) $v = v_k^{(1)}$;
(2) $v_k \neq v_{k-1}^{(1)}$.

In particular, $wv^{[\infty]}$ is reduced if (1) and (2) are satisfied.

Proof. Suppose (w, v) is compatible, arising from $X \in \Omega$ with ASW factorization $X = \prod_{j \geq 1} N_j$. As before, let l be minimal such that $\ell(v(N_l + k)) = \ell(v(N_l))$ for every $k \geq 0$. We argue that $w = \prod_{j=1}^{l} v(N_j)$ is the combinatorial ASW factorization of w. Then both (1) and (2) follow from Lemma 7.13. Suppose for some j that $v(N_j)$ is not the maximal cyclically increasing element that can be factored from $v(N_j)v(N_{j+1}) \cdots v(N_l)$. By applying braid and commutation moves to $N_j N_{j+1} \cdots N_l$ we see that N_j is not the maximal curl that can be factored out from $N_j N_{j+1} \ldots N_l$, which contradicts the main property of ASW factorization. Thus by definition $w = v(N_1)v(N_2) \cdots v(N_l)$ is the combinatorial ASW factorization of w.

Now suppose that (w, v) satisfy the conditions (1) and (2) of the Proposition. Let $v' = v_k$, so that $v = (v')^{(1)}$. Let $c(v')$ be the Coxeter element in which s_i precedes s_{i+1} if and only if s_i is contained in v'. Then there is a length additive factorization $c(v') = v'u$. Furthermore, it is easy to see that $c((v')^{(1)}) = uv'$. Let $Z \in A_{c^\infty}$ (see Section 6.3 and Proposition 6.7). Now perform the ASW factorization of Z to get $Z = \prod_{i=1}^{\infty} N_i$. Then $v(N_1) = v'$, and the argument in the proof of Proposition 6.9 shows that $\prod_{i=2}^{\infty} N_i \in A_{c((v')^{(1)})^\infty}$. Repeating, we deduce that $v(N_i) = v^{(i)}$. Thus Z is in the (v', v) ASW-cell.

Now let $Y = N'_1 N'_2 \cdots N'_{k-1}$, where N'_i is any degenerate curl satisfying $v(N'_i) = v_i$. We claim that $X = YZ$ is in the (w, v) ASW-cell. Let $X_r = N'_r N'_{r+1} \cdots N'_{k-1} Z$. We shall show by decreasing induction that $N'_r = N(\epsilon_1(Y_r), \ldots, \epsilon_n(Y_r))$. We already know the base case $Y_k = Z$. The inductive step follows from Proposition 7.14(2) and Lemma 6.4. It follows that ASW-factorization applied to X extracts the curls $N'_1, N'_2, \ldots$, and that X is in the (w, v) ASW-cell. In particular, we deduce from Lemma 7.6 that $wv^{[\infty]}$ is reduced. $\square$

Example 7.4. Let $n = 4$, $w = s_1 s_2 s_3 s_0 s_2 s_1 s_3 s_2$, $v = s_1$. Then $v_1 = s_1 s_2 s_3$, $v_2 = s_0 s_2$, $v_3 = s_1 s_3$, $v_4 = s_2$ and the pair (w, v) is compatible.

Remark 7.5. There is a whirl version of ASW factorization, with whirls replacing curls, abd μ_i's replacing ϵ_i's. As a result one obtains a factorization of X into maximal whirl factors. All the properties of ASW factorization have an analogous form that holds for this whirl-ASW factorization. For example, the analogs of Theorem 7.5, Lemma 7.9, Proposition 7.14 and Proposition 7.15 hold, where in the case of the latter two one needs to change the definition of $v^{(k)}$ to $v^{(k)} = \prod_j s_{i_j+k}$, and cyclically decreasing permutations occur instead of cyclically increasing ones.

8. Totally positive exchange lemma

8.1. Statement of Lemma, and proof of Theorem 5.5.

Theorem 8.1 (Totally positive exchange lemma)**.** *Suppose*

$$X = e_r(a)e_{i_1}(a_1)\cdots e_{i_\ell}(a_\ell) = e_{i_1}(a'_1)\cdots e_{i_\ell}(a'_\ell)e_j(a')$$

are reduced products of Chevalley generators such that all parameters are positive. For each $m \leq \ell$ and each $x \in \mathbb{Z}/n\mathbb{Z}$ define $S = \{s \leq m \mid i_s = x\}$. Then

$$\sum_{s\in S} a'_{i_s} \leq \begin{cases} \sum_{s\in S} a_{i_s} & \text{if } x \neq r, \\ a + \sum_{s\in S} a_{i_s} & \text{if } x = r. \end{cases} \tag{5}$$

Using the totally positive exchange lemma, we now prove Theorem 5.5.

Proof of Theorem 5.5. Define $\mathbf{a}' = R_{\mathbf{i}}^{\mathbf{j}}(\mathbf{a})$. Let $X = e_{\mathbf{i}}(\mathbf{a})$, and $X^{(k)} = e_{i_1}(a_1)\cdots e_{i_k}(a_k)$. Let $Y = e_{\mathbf{j}}(\mathbf{a}')$ and define $Y^{(k)}$ similarly. By the definition of braid limit, it follows that for each $k > 0$, there is k' so that $Y^{(k)} < X^{(k')} < X$ entry-wise. We need to show that $X^{(k)} < Y$ for each k.

Let $Z^{(k)}$ be such that $Y^{(k)}Z^{(k)} = X$. It is clear from the the definition of braid limit that $Z^{(k)}$ is TNN (in fact, $Z^{(k)} \in \Omega$). Let $Z = Y^{-1}X$. Then $Z = \lim_{k\to\infty} Z^{(k)}$ and by [LPI, Lemma 2.4], Z is TNN. We shall show that the entries of Z directly above the diagonal (that is $z_{i,i+1}$) vanish, which in turn implies that Z is the identity, or equivalently, $X = Y$.

Fix $i \in \mathbb{Z}/n\mathbb{Z}$, and write $\chi(X) = x_{i,i+1}$ for any $X \in U$. We now show that for each $k > 0$, we have $\chi(X^{(k)}) < \chi(Y)$. Since $\chi(X) = \chi(Z) + \chi(Y)$, this will prove that $X = Y$. Note that $\chi(e_{i_1}(b_1)\cdots e_{i_r}(b_r)) = \sum_{s:i_s=i} b_s$. If X is a (possibly infinite) product of Chevalley generators, we let $\chi_r(X)$ be χ of the product of the first r generators in X.

By Propositions 5.3 and 4.9, we may assume that the braid limit $\mathbf{i} \to \mathbf{j}$ is obtained by infinite exchange. Suppose that the generators $i_1, i_2, \ldots, i_k$ are all"crossed out" by the r-th step in infinite exchange. We let s be the rightmost generator of $\mathbf{i}$ to be crossed out in the first r-steps. Define k_j for $j = 0, 1, \ldots, s$ as follows: set $k_0 = k$ and let $k_j = k_{j-1}+1$ if the generator crossed out in the j-th step of infinite exchange is to the right of i_k, and $k_j = k_{j-1}$ otherwise. Then $k_r = r$.

Let $A, B, C, \ldots$ be the matrices obtained from X by performing one, two, three, and so on, iterations of infinite exchange. Let V be the matrix obtained after r iterations of infinite exchange. Note that the first r factors of V are the same as the first r factors of Y.

Using Theorem 8.1,

$$\chi(X^{(k)}) = \chi_k(X) \leq \chi_{k_1}(A) \leq \chi_{k_2}(B) \leq \cdots \leq \chi_{k_r}(V) = \chi_r(V) = \chi_r(Y) < \chi(Y).$$

□

We shall give two proofs of Theorem 8.1. The first proof relies on the machinery of the Berenstein-Zelevinsky Chamber Ansatz [BZ], and is a direct calculation of the two sides of (5). The second proof is less direct, but significantly shorter.

8.2. First Proof of Theorem 8.1.

We may assume $n > 2$ for otherwise the statement is vacuous. Let us call a subset $S \subset \mathbb{Z}$ *a-nice* if it occurs as $S = w(\mathbb{Z}_{\leq a})$ for some $w \in \tilde{W}$. We say that $S \subset \mathbb{Z}$ is *nice* if it is 0-nice. Clearly a subset can be a-nice for at most one a. If $a = a' + bn$ we will often identify an a-nice subset I with the a'-nice subset $I - bn$.

Lemma 8.2. *Let $I \subset \mathbb{Z}$ and $J = \mathbb{Z} \setminus I$. Then $I \subset \mathbb{Z}$ is nice if and only if $I - n \subset I$, $J + n \subset J$ and $|I \cap \mathbb{Z}_{>0}| = |J \cap \mathbb{Z}_{\leq 0}|$ is finite.*

A quadruple $D = (I, i, j, J)$ is *nice* if it can be obtained from $w \in \tilde{W}$, by setting

$$D(w) = (I = w(\mathbb{Z}_{\leq -1}), w(0), w(1), J = w(\mathbb{Z}_{>1})).$$

In particular, I is -1-nice, $I \cup \{i\}$ is 0-nice, and $I \cup \{i, j\}$ is 1-nice.

We will write Ki to denote the set $K \cup \{i\}$, and say that Ki is nice, if $i \notin K$, and both $K \cup \{i\}$ and K are nice. Note that this implies that i is maximal in its residue class modulo n, within $K \cup \{i\}$. Similarly, we shall use notation Kij, $Kijk$, and so on. In this latter notation, we will always assume that $i < j < k$ have distinct residues modulo n.

8.3. **Berenstein-Zelevinsky Chamber Ansatz.** We recall some definitions and results from [BZ]. The results in [BZ] are stated for finite-dimensional algebraic groups, but as remarked there, can be extended to the Kac-Moody case. In particular, they apply to $U_{\geq 0}^{\mathrm{pol}}$. Our notations differ from theirs by $w \leftrightarrow w^{-1}$.

A *chamber weight* is an extremal weight of a fundamental representation. Every chamber weight is of the form $w \cdot \omega_a$ where $w \in \tilde{W}$, and ω_a is a fundamental weight of $\widehat{sl}(n)$. Chamber weights in the orbit of ω_a are in bijection with a-nice subsets, via $w \cdot \omega_a \leftrightarrow w(\mathbb{Z}_{\leq a})$. Hereon, we identify chamber weights with nice subsets.

Let $v \in \tilde{W}$. Recall that we denote by $\mathrm{Inv}(v)$ the set of inversions of v. Let I be a nice-subset. An inversion of I is a positive root $\alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$ such that $j \in I$ but $i \notin I$. We denote by $\mathrm{Inv}(I)$ the set of inversions of I. For convenience, in the following we will identify positive roots with pairs $i < j$ (with different residues modulo n).

We define, following [BZ], the set E^v of v-chamber weights by

$$E^v = \{w(\mathbb{Z}_{\leq a}) \mid w \leq v\}.$$

If $I \in E^v$, we say I is v-nice.

Proposition 8.3 ([BZ, Proposition 2.8])**.** *We have $I \in E^v$ if and only if* $\mathrm{Inv}(I) \subset \mathrm{Inv}(v)$.

Example 8.1. Let $n = 3$ and let $w = s_1 s_2 s_1 s_0$. The step-by-step computation of $w(\mathbb{Z}_{\leq 0})$ proceeds as follows: $s_0(\mathbb{Z}_{\leq 0}) = \{1\} \cup \mathbb{Z}_{\leq -1}$, $s_1 s_0(\mathbb{Z}_{\leq 0}) = \{2\} \cup \mathbb{Z}_{\leq -1}$, $s_2 s_1 s_0(\mathbb{Z}_{\leq 0}) = \{0, 3\} \cup \mathbb{Z}_{\leq -2}$, $s_1 s_2 s_1 s_0(\mathbb{Z}_{\leq 0}) = \{-1, 0, 3\} \cup \mathbb{Z}_{\leq -3}$. The inversions of $I = \{-1, 0, 3\} \cup \mathbb{Z}_{\leq -3}$ are $\mathrm{Inv}(I) = \{(-2, -1), (-2, 0), (-2, 3), (1, 3), (2, 3)\}$. This set is identified with the set of roots $\mathrm{Inv}(I) = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \delta\}$, where pairs $(-2, 0)$ and $(1, 3)$ correspond to the same root $\alpha_1 + \alpha_2$. This set is contained in the set of inversions of w (in fact, equal to it). Therefore for any $w \leq v$ we have $\mathrm{Inv}(I) \subset \mathrm{Inv}(v)$.

We suppose that $v \in \tilde{W}$ has been fixed. Let

$$M_\bullet = (M_I)_{I \in E^v}$$

be a collection of positive real numbers satisfying the relations [BZ, (4.5)]

$$M_{ws_a\mathbb{Z}_{\leq a}} M_{ws_{a+1}\mathbb{Z}_{\leq a+1}} = M_{w\mathbb{Z}_{\leq a}} M_{ws_a s_{a+1}\mathbb{Z}_{\leq a+1}} + M_{ws_{a+1}s_a\mathbb{Z}_{\leq a}} M_{w\mathbb{Z}_{\leq a+1}} \tag{6}$$

for each $w \in \tilde{W}$ such that $ws_a s_{a+1} s_a$ is length-additive, and such that all mentioned chamber weights lie in E^v.

Suppose $\mathbf{i} = i_1 i_2 \cdots i_\ell$ is a reduced word for v, and $X = e_{i_1}(a_1) \cdots e_{i_\ell}(a_\ell) \in U_{\geq 0}^{\mathrm{pol}}$, where the a_i are positive parameters. For each reduced word $\mathbf{j} = j_1 j_2 \cdots j_\ell$ of v, we define parameters $\mathbf{a}^{\mathbf{j}} = (a_1^{\mathbf{j}}, a_2^{\mathbf{j}}, \ldots, a_\ell^{\mathbf{j}}) = R_{\mathbf{i}}^{\mathbf{j}}(\mathbf{a})$ (see Corollary 3.5).

Given a nice quadruple $D = (I, i, j, J)$, we now define the positive numbers

$$M(D) = \frac{M_{I\cup\{i,j\}} M_I}{M_{I\cup\{i\}} M_{I\cup\{j\}}}.$$

If $D = (I, i, j, J)$, then J is determined by I, i, j, so we shall often write $M(I, i, j)$ instead of $M(I, i, j, J)$ for $M(D)$. We say that D, or $M(D)$, is v-nice if all the indexing subsets in this formula are v-nice. By abuse of notation, we write $M(w) := M(D(w))$. Note that if both $w \le v$ and $ws_0 \le v$, then $M(w)$ is v-nice.

Theorem 8.4 ([BZ]). *There is a bijection between the collections* $\{a_k^{\mathbf{j}}\}$ *as* $\mathbf{j}$ *varies over all reduced words of* v*, and collections of positive real numbers* $M_\bullet = (M_I)_{I\in E^v}$ *satisfying* (6)*, given by*

$$a_k^{\mathbf{j}} = M(D(s_{j_1} s_{j_2} \cdots s_{j_{k-1}})).$$

8.4. **Relations for M_I and $M(D)$.** For an arbitrary subset $I \subset \mathbb{Z}$, we say that M_I is a-nice, if I is a-nice. We say that M_I is nice if it is a-nice for some a. We first reinterpret Proposition 8.3 in a more explicit manner (see also [BFZ]).

Lemma 8.5. *Suppose $i < j < k$ have different residues and K is nice. Then if one of the three pairs (Kik, Kj), (Kij, Kk), and (Kjk, Ki) consist of nice subsets, then all three do. If two of the three pairs consist of v-nice subsets, then so is the third one. We then have*

$$M_{Kik} M_{Kj} = M_{Kij} M_{Kk} + M_{Kjk} M_{Ki} \tag{7}$$

Proof. The first statement is straightforward, and indeed implies that $Kijk$ is nice. To check the second statement, we note that there are three possible inversions amongst $i < j < k$, and that the v-niceness of each of the three pairs (Kik, Kj), (Kij, Kk), and (Kjk, Ki) imply that $\mathrm{Inv}(v)$ contains two of these inversions. It is easy to check that if two of the three pairs are v-nice, then $\mathrm{Inv}(v)$ contains all three inversions.

To obtain (7), apply (6) to some $w \in \tilde{W}$ satisfying $w(a) = i$, $w(a+1) = j$, $w(a+2) = k$, and $w(\mathbb{Z}_{<a}) = K$. $\square$

Lemma 8.6. *Suppose $i < j < k$ have different residues, such that $Kijk$ is nice. Then assuming v-niceness, we have*

$$M(Kj, i, k) = M(Ki, j, k) + M(Kk, i, j)$$
$$M(Ki, k, j) = M(Kj, k, i) + M(Kk, i, j).$$

Furthermore, in either equation, if two terms are known to be v-nice, then the third term is as well.

Proof. Using (7), we calculate

$$\begin{aligned} M(Kj, i, k) - M(Ki, j, k) &= \frac{M_{Kijk} M_{Kj}}{M_{Kij} M_{Kjk}} - \frac{M_{Kijk} M_{Ki}}{M_{Kij} M_{Kik}} \\ &= \frac{M_{Kijk}}{M_{Kij} M_{Kik} M_{Kjk}} (M_{Kj} M_{Kik} - M_{Ki} M_{Kjk}) \\ &= M(Kk, i, j). \end{aligned}$$

The second statement is similar. The last statement follows from Lemma 8.5. $\square$

Lemma 8.7. *Suppose $i < j < k < l$ have different residues, such that $Kijkl$ is nice. Then assuming v-niceness,*

$$M(Kl, j, k) - M(Ki, j, k) = M(Kk, i, l) - M(Kj, i, l)$$

and

$$M(Kk, i, j) - M(Kl, i, j) = M(Ki, k, l) - M(Kj, k, l).$$

Furthermore, if the terms on the same side of either equation are v-nice, then all six inversions amongst $\{i, j, k, l\}$ are contained in $\mathrm{Inv}(v)$.

Proof. The last statement is checked directly, and implies that all the subsets in the following calculations are v-nice.

We first prove the first equation, omitting K from the notation for simplicity. In the following we use (7) repeatedly.

$$\begin{aligned}
&\frac{M_{ijl}M_j}{M_{ij}M_{jl}} - \frac{M_{ijk}M_i}{M_{ij}M_{ik}} \\
&= \frac{M_{ijl}(M_{ij}M_k + M_{jk}M_i) - M_{ijk}M_{jl}M_i}{M_{ij}M_{jl}M_{ik}} \\
&= \frac{M_{ijl}M_k}{M_{jl}M_{ik}} + \frac{M_i(M_{ijl}M_{kj} - M_{ijk}M_{jl})}{M_{ij}M_{jl}M_{ik}} \\
&= \frac{M_{ijl}M_k + M_{jkl}M_i}{M_{jl}M_{ik}} \\
&= \frac{M_{ijl}M_k}{M_{jl}M_{ik}} + \frac{M_{jkl}(M_{il}M_k - M_l M_{ik})}{M_{ik}M_{kl}M_{jl}} \\
&= \frac{M_k(M_{ijl}M_{kl} + M_{jkl}M_{il}) - M_{jkl}M_l M_{ik}}{M_{ik}M_{kl}M_{jl}} \\
&= \frac{M_{ikl}M_k}{M_{ik}M_{kl}} - \frac{M_{jkl}M_l}{M_{jl}M_{kl}}
\end{aligned}$$

For the second equation, we calculate

$$\begin{aligned}
&\frac{M_{ijk}M_k}{M_{ik}M_{jk}} - \frac{M_{ikl}M_i}{M_{ik}M_{il}} \\
&= \frac{M_{ijk}M_{ik}M_l + M_{ijk}M_{kl}M_i - M_{ikl}M_{jk}M_i}{M_{ik}M_{jk}M_{il}} \\
&= \frac{M_{ijk}M_l - M_{jkl}M_i}{M_{jk}M_{il}} \\
&= \frac{M_l M_{ijk}M_{jl} + M_l M_{jkl}M_{ij} - M_{jkl}M_j M_{il}}{M_{il}M_{jk}M_{jl}} \\
&= \frac{M_{ijl}M_l}{M_{il}M_{jl}} - \frac{M_{jkl}M_j}{M_{jk}M_{jl}}
\end{aligned}$$

$\square$

8.5. **Explicit formula for difference of sum of parameters.** Let $v = s_{i_1}s_{i_2}\cdots s_{i_{\ell(v)}}$, and $w = s_{i_1}s_{i_2}\cdots s_{i_\ell}$. Let

$$X = e_{\bar{r}}(a)e_{i_1}(a_1)\cdots e_{i_{\ell(v)}}(a_{\ell(v)}) = e_{i_1}(a'_1)\cdots e_{i_{\ell(v)}}(a'_{\ell(v)})e_j(a')$$

as in Theorem 8.1, and without loss of generality we assume $j = 0$. We write $\bar{r} \in \mathbb{Z}/n\mathbb{Z}$ instead of r as we shall use the latter for a specific representative of $\bar{r}$. Let M_I for $I \in E^v$ be defined via Theorem 8.4, using the parameters $a_k^{\mathbf{j}}$ for X.

For any $u \leq v$, define $N(u)$ as follows. Pick a reduced factorization $u = s_{j_1}\cdots s_{j_k}$ and write $X = e_{j_1}(b_1)\cdots e_{j_k}(b_k)Y$ where Y is in $E^{u^{-1}v}$. Then $N(u) = \sum_{s|j_s=0} b_{i_s}$. Thus to prove the theorem we must show that for every w such that $\ell(s_rw) > \ell(w)$ and $w, s_rw \leq v$, we have $N(s_rw) \geq N(w)$.

To prove the theorem, we may further assume that $i_\ell = 0$, and we let $w' = ws_0 < w$. Unless otherwise specified, $r \in \mathbb{Z}$ is the maximal representative of $\bar{r}$ such that $r \in w(\mathbb{Z}_{\leq 1})$.

Lemma 8.8. *Suppose $v, w \in \tilde{W}$ and $r \in \mathbb{Z}$ is such that $\ell(s_rw) > \ell(w)$. If $w \leq v$ and $s_rw \leq v$ then $n > w^{-1}(r+1) - w^{-1}(r) > 0$.*

Proof. The inequality $w^{-1}(r+1) - w^{-1}(r) > 0$ follows from $\ell(s_rw) > \ell(w)$. Suppose $w^{-1}(r+1) - w^{-1}(r) > n$. Then $(r+1 < r+n)$ is an inversion in w. But then $(r+1 < r+n)$ will also be an inversion in v. This is impossible as $(r < r+1)$ is an inversion in s_rw, which means it is also an inversion in v. □

Note that $w \leq v$ and $s_rw \leq v$ implies that, w-nice and s_rw-nice subsets are also v-nice.

Lemma 8.9. *Suppose that we are in the situation of Lemma 8.8. Let $D(w') = (I, i, j, J)$.*

(1) *If $r, r+1$ both lie in I, then $N(s_rw) - N(w) = 0$.*
(2) *If ($i = r$ and $r+1 \in J$) or ($j = r+1$ and $r \in I$), then $N(s_rw) - N(w) = 0$.*
(3) *Otherwise $N(s_rw) - N(w) = M(D'(w'))$ where $D'(w') = (I', i', j', J')$ is obtained from $D(w') = (I, i, j, J)$ by setting $i' = r$, $j' = r+1$ and*

$$I' = \begin{cases} I \setminus \{r\} \cup \{j\} & \text{if } r \in I \text{ but } r+1 \notin I \\ I & \text{if } \{r, r+1\} \cap I = \emptyset. \end{cases}$$

Example 8.2. Let $n = 4$, $v = s_0s_1s_2s_1s_3s_0s_1s_3$, $j = 0$. Then $vs_0 = s_1v$ so that $\bar{r} = \bar{1}$. Let $w = s_0s_1s_2s_1s_3s_0$, and thus $w' = s_0s_1s_2s_1s_3$. We have an equality

$$e_1(a)e_0(a_1)e_1(a_2)e_2(a_3)a_1(a_4)e_3(a_5)e_0(a_6)e_1(a_7)e_3(a_8) =$$

$$= e_0(a'_1)e_1(a'_2)e_2(a'_3)a_1(a'_4)e_3(a'_5)e_0(a'_6)e_1(a'_7)e_3(a'_8)e_0(a'_9)$$

for some positive parameters, and we are interested in the value of $N(s_1w) - N(w) = a_1 + a_6 - a'_1 - a'_6$. We compute that $w(\mathbb{Z}_{\leq 1}) = \{1, 3\} \cup \mathbb{Z}_{\leq -1}$, from which we find $r = 1$. We further compute $D(w') = (I, -4, 3, J)$, where $I = \{-3, -2, -1, 1\} \cup \mathbb{Z}_{\leq -5}$, $J = \{0, 2\} \cup \mathbb{Z}_{\geq 4}$. Then we are in case (3), and furthermore in situation $r \in I$ but $r+1 \notin I$. This allows us to find $D'(w') = (I', 1, 2, J')$ where $I' = \{-3, -2, -1, 3\} \cup \mathbb{Z}_{\leq -5}$, $J' = \{-4, 0\} \cup \mathbb{Z}_{\geq 4}$. Then $M(D'(w'))$ is the needed manifestly positive value of $N(s_1w) - N(w)$.

It may not be clear that $D'(w')$ is a v-nice set, but this will follow from our calculations.

Proof of Theorem 8.1. According to Lemma 8.9, the difference $N(s_rw) - N(w)$ that we are interested in is manifestly nonnegative. □

To prove the lemma we proceed by induction on the number of times 0 occurs in $\mathbf{i}$. By assumption 0 occurs at least once. In the following calculations, all the nice subsets that occur will in fact be v-nice, and this will follow from the last statements of Lemmata 8.6 and 8.7; we will not mention this explicitly.

8.6. **Base Case.** Suppose 0 occurs once in $\mathbf{i}$. Then by Theorem 8.4, $N(w) = M(w')$. Let $(I, i, j, J) = D(w')$. We note that our assumption implies that $I \cup \{i\} = \mathbb{Z}_{\leq 0}$.

Case $r < 0$: we have $N(s_r w) = M(s_r w')$. Suppose first that $r, r+1 \in I$. Then $M(w') = M(s_r w')$, so that $N(s_r w) - N(w) = 0$, agreeing with Lemma 8.9(1). Otherwise, we must have $i = r+1$. Let $K = I - \{r\}$. Then using Lemma 8.6,

$$N(s_r w) - N(w) = M(K(r+1), r, j) - M(Kr, r+1, j) = M(Kj, r, r+1)$$

which is $M(D'(w'))$, as required.

Case $r = 0$: first note that we cannot have $i = 0$ and $j = 1$, for this would mean that $s_0 w' s_0$ is not length-additive. Suppose first that $0 \in I$ and $1 \in J$. Then by Lemma 8.8, $0, 1, i, j$ all have distinct residues modulo n. Let $K = I \setminus \{0\}$ and $L = J \setminus \{1\}$. Then using Lemma 8.7, we have $N(s_r w) - N(w) = M(Ki, 0, 1) + M(K1, i, j) - M(K0, i, j) = M(Kj, 0, 1)$ as required. Suppose that $i = 0$ and $1 \in J$. Then using (7), we have $N(s_r w) - N(w) = M(I, 0, 1) + M(I, 1, j) - M(I, 0, j) = 0$, agreeing with Lemma 8.9(2). The last case $0 \in I$ and $j = 1$ is similar.

Case $r > 0$: this is similar to $r < 0$.

8.7. **Inductive Step.** Now suppose that the letter 0 occurs more than once in $\mathbf{i}$. Let $u = s_{i_1} s_{i_2} \cdots s_{i_{\ell'}}$, where $i_{\ell'} = 0$ and $i_{\ell'+1}, i_{\ell'+2}, \ldots, i_{\ell-1}$ are all distinct from 0. We shall assume that Lemma 8.9 is known to hold for u. Let us first compare $D(u') = (A, a, b, B)$ with $D(w') = (I, i, j, J)$. Since $s_{i_{\ell'}} = s_0$, we have $I \cup \{i\} = A \cup \{b\}$ and $J \cup \{j\} = B \cup \{a\}$. Furthermore, one notes that we cannot have both $i = b$ and $j = a$. Also one cannot have both $i = r$ and $j = r+1$.

We make two preparatory remarks:

1. We shall use Lemma 8.7 repeatedly in the following, where $\{i, j, k, l\}$ of the Lemma will usually be $\{i, j, r, r+1\}$. The assumption that $Kijkl$ is nice will follow from the fact that the positions of $i, j, r, r+1$ in w are within a "window" of size n.
2. In the beginning we chose r to be the maximal representative of $\bar{r}$ in $w(\mathbb{Z}_{\leq 1})$. This choice of r is also the maximal representative of $\bar{r}$ for u, except in one case: when $a = r+n$, $b \neq r+n+1$, $r+n \in J$, and $\{r, r+1\} \subset I$.

By Theorem 8.4, we have

$$N(s_r w) - N(w) = N(s_r u) - N(u) + M(s_r w') - M(w'). \tag{8}$$

Case 1: $\{r, r+1\} \subset I$. We have $M(w') = M(s_r w')$. If $\{r, r+1\} \subset A$ as well, then by induction and (8) we have $N(s_r w) - N(w) = N(s_r u) - N(u) + M(s_r w') - M(w') = 0$ by Lemma 8.9(1). Otherwise $r \in A$ and $r+1 = b$ (this includes the case $a = r+n$). Then $N(s_r w) - N(w) = 0$ as well by Lemma 8.9(2). In either case, we have verified that $N(s_r w) - N(w)$ agrees with Lemma 8.9(1).

Case 2: $r \in I$ and $i = r+1$. We have two possibilities for (A, a, b, B): (a) $\{r, r+1\} \subset A$, (b) $r \in A$ and $b = r+1$. In either case, the inductive hypothesis says that $N(s_r u') - N(u') = 0$. We have $M(w') = M(Kr, r+1, j)$ and $M(s_r w') = M(K(r+1), r, j)$ where

$K = I \backslash \{r\}$. By Lemma 8.6 and (8), $N(s_r w) - N(w) = M(K(r+1), r, j) - M(Kr, r+1, j) = M(Kj, r, r+1) = M(D'(w'))$, as required.

Case 3: $r+1 \in J$ and $j = r$. Same as Case 2.

Case 4: $r \in I$ and $j = r+1$. By length-additivity of $w = w's_0$, we have $i < r+1$. Let $K = I \setminus \{r\}$. We have three possibilities for (A, a, b, B): (a) $r \in A$ and $r+1 \in B$, (b) $r \in A$ and $a = r+1$, (c) $r+1 \in B$ and $b = r$. In all three cases, one has $N(s_r u) - N(u) = M(Ki, r, r+1)$. One calculates using Lemma 8.6 and (8) that

$$N(s_r w) - N(w) = M(Ki, r, r+1) + M(K(r+1), i, r) - M(Kr, i, r+1) = 0$$

agreeing with Lemma 8.7(2).

Case 5: $r+1 \in J$ and $i = r$. Same as Case 4.

Case 6: $r \in I$ and $r+1 \in J$. Let $K = I \setminus \{r\}$.

We have three possibilities for (A, a, b, B): (a) $r \in A$ and $r+1 \in B$, (b) $r+1 \in B$ and $b = r$, and (c) $r \in A$ and $a = r+1$. In all three cases, we have $N(s_r u) - N(u) = M(Ki, r, r+1)$ and calculate using (8)

$$\begin{aligned} N(s_r w) - N(w) &= M(Ki, r, r+1) + M(K(r+1), i, j) - M(Kr, i, j) \\ &= M(Kj, r, r+1) \end{aligned}$$

using the two forms of Lemma 8.7, depending on whether $i < j < r < r+1$, $i < r < r+1 < j$, or $r < r+1 < i < j$. This agrees with Lemma 8.9(3).

This completes the proof of Lemma 8.9.

8.8. **Second Proof of Theorem 8.1.** We use the notation for w and v, and $N(u)$ of Section 8.5. Without loss of generality we can assume $s_x = s_0$, as before.

Lemma 8.10. *It suffices to prove Theorem 8.1 in the case $w = s_{i_1} \dots s_{i_\ell}$ has a single right descent s_0.*

Proof. Write $w = uy$, where $\ell(u) + \ell(y) = \ell(w)$ and $y \in W$. Then $N(w) = N(u)$ and $N(s_r w) = N(s_r u)$, so we may replace w by u. □

Lemma 8.11. *Suppose w has a unique right descent s_0, and that $s_r w > w$ and the join $v' = w \vee s_r w$ exists in weak order. Then*

1. *if $w^{-1}(r+1) = w^{-1}(r) + 1 = l+1$, then $l \neq 0$ modulo n and $v' = ws_l$;*
2. *if $w^{-1}(r+1) = k$, $w^{-1}(r) = l$ and $k > l+1$, then $[l, k]$ contains a unique number m of residue 0 modulo n and $v' = ws_l s_{l+1} \dots s_{m-1} s_{k-1} s_{k-2} \dots s_{m+1} s_m$.*

Proof. By Lemma 8.8, we have $w^{-1}(r) < w^{-1}(r+1) < w^{-1}(r) + n$.

If $w(r+1) = w(r) + 1 = l+1$ then $ws_l = s_r w$ and thus $s_r w > w$. This implies that $s_r w = w \vee s_r w$.

Assume now $w^{-1}(r+1) = k$, $w^{-1}(r) = l$, and $k > l+1$. Then the sequence $r = w(l), w(l+1), \dots, w(k) = r+1$ cannot be increasing, and thus has at least one descent. Since the only right descent of w is s_0, the first claim follows. Furthermore, it has to be the case that

$$r = w(l) < w(l+1) < \dots < w(m) > w(m+1) < w(m+2) < \dots < w(k) = r+1.$$

Then we see that $v' = ws_l s_{l+1} \dots s_{m-1} s_{k-1} s_{k-2} \dots s_{m+1} s_m$ is reduced since at each step an inversion is created, resulting in

$$v'([l, k]) = w(l+1), \cdots, w(m), r+1, r, w(m+1), w(m+2), \cdots, w(k-1).$$

One also has

$$ws_ls_{l+1}\dots s_{m-1}s_{k-1}s_{k-2}\dots s_{m+1}s_m = s_rws_ls_{l+1}\dots s_{m-1}s_{k-1}s_{k-2}\dots s_1$$

and so $v' > w, s_rw$ in weak order. It remains to argue that v' is the minimal upper bound. The inversion set $\mathrm{Inv}(w \vee s_rw)$ contains the inversion of $(r < r+1)$ in s_rw, and the inversions $\{(r+1 < w(l+1)), \dots, (r+1 < w(m))\}$ together with $\{(w(m+1) < r), \dots, (w(k-1) < r)\}$ in w. By biconvexity (see Section 4.1), $\mathrm{Inv}(w \vee s_rw)$ must also contain $\{(r < w(l+1), \dots, (r < w(m))\}$ and $\{(w(m+1) < r+1), \dots, (w(k-1) < r+1)\}$. These extra inversions are present in $\mathrm{Inv}(v)$ and the number of extra inversions is exactly $\ell(v) - \ell(s_rw)$. Thus $v' = w \vee s_rw$. □

Proof of Theorem 8.1. By Lemma 8.10 we can assume w has a single right descent s_0. Since $v > w, s_rw$ we know that w and s_rw have a join v in weak order. Furthermore, the join v' is given by Lemma 8.11. It remains to note that $N(s_rw) = N(v')$ since $v' = s_rwy$ with $\ell(s_rw) + \ell(y) = \ell(v')$ and $y \in W$. On the other hand, $N(w) \leq N(v')$ as well. Thus $N(s_rw) \geq N(w)$, as desired. □

Example 8.3. In the situation of Example 8.2 the join of $w = s_0s_1s_2s_1s_3s_0$ and s_1w is exactly $s_0s_1s_2s_1s_3s_0s_1s_3s_0 = s_1s_0s_1s_2s_1s_3s_0s_1s_3$, which shows that $N(s_1w) - N(w) = a_1 + a_6 - a_1' - a_6' = a_9'$ is manifestly positive.

9. Greedy factorizations

Suppose $X \in U_{\geq 0}$. A factorization $X = e_i(a)X'$ with $a \geq 0$ and $X' \in U_{\geq 0}$ is called *greedy* if $e_i(-a')X$ is not TNN for $a' > a$. Since limits of TNN matrices are TNN [LPI], we can equivalently say that $X = e_i(a)X'$ is greedy if

$$a = \sup(a' \geq 0 \mid e_i(-a')X \in U_{\geq 0})$$

where the right hand side is always equal to $\max(a' \geq 0 \mid e_i(-a')X \in U_{\geq 0})$.

More generally, a factorization $X = e_{i_1}(a_1)e_{i_2}(a_2)\cdots e_{i_r}(a_r)X'$ is called greedy if the factorization $e_{i_k}(a_k)\left(e_{i_{k+1}}(a_{k+1})\cdots e_{i_r}(a_r)X'\right)$ is greedy for every $k \in [1, r]$. A factorization $X = e_{i_1}(a_1)e_{i_2}(a_2)\cdots$ is called greedy if the factorization $e_{i_k}(a_k)\left(e_{i_{k+1}}(a_{k+1})\cdots\right)$ is greedy for every $k \geq 1$.

As was shown in Proposition 6.2, the maps $e_{\mathbf{i}}$ are not injective in general. Restricting to greedy factorizations fixes this problem to some extent: for each X and infinite reduced word, there is at most one greedy factorization $X = e_{\mathbf{i}}(\mathbf{a})$.

Proposition 9.1. *Let $X \in \Omega$. Then X has a complete greedy factorization.*

Proof. By Theorem 7.12, we may factor (infinitely many) Chevalley generators from X greedily in any manner, and the resulting product will be equal to X. □

Thus greedy factorizations do "cover" Ω.

9.1. **Minor ratios for greedy parameters.** If $I = \{i_1 < i_2 < \cdots < i_l\}$ and $J = \{j_1 < j_2 < \cdots < j_k\}$ are two sets of integers of the finite cardinality, we say that I is less than or equal to J, written $I \leq J$, if $i_r \leq j_r$ for each $r \in [1, \min(k, l)]$. We say that I is much smaller than J and write $I \ll J$ if $i_r < j_r$ for each $r \in [1, \min(k, l)]$

One can use limits of minor ratios to factor an element of Ω greedily. Let $I = i_1 < i_2 < \ldots < i_l$ and $I' = i_1' < i_2' < \ldots < i_l'$ be two sets of row indices such that $I \leq I'$. Let

$h = \min(i_1, i'_1)$ and let $I_k = I \cup \{h-k, \ldots, h-1\}$ and $I'_k = I' \cup \{h-k, \ldots, h-1\}$. In particular, one has $I_0 = I$ and $I'_0 = I'$.

The following Lemma will be proved in Section 9.3.

Lemma 9.2. *Let $X \in U_{>0}$ be totally positive, and $I \leq I'$ be fixed. Let J_k be a sequence of column sets such that $|J_k| = k + l$ and $J_{k-1} \ll J_k$. The limit*

$$\ell = \lim_{k\to\infty} \frac{\Delta_{I_k,J_k}(X)}{\Delta_{I'_k,J_k}(X)}$$

exists and does not depend on the choice of sequence J_k. Furthermore, $\ell \leq \frac{\Delta_{I_k,J_k}(X)}{\Delta_{I'_k,J_k}(X)}$ for any J_k.

Lemma 9.2 allows us to introduce the notation

$$\frac{X_{[\ldots I]}}{X_{[\ldots I']}} = \lim_{k\to\infty} \frac{\Delta_{I_k,J_k}(X)}{\Delta_{I'_k,J_k}(X)}$$

that does not include $\{J_k\}$ in it. The following theorem is the key motivation for looking at this kind of minor ratio limits.

Proposition 9.3. *Suppose $X \in U_{>0}$ is totally positive. Let $a = \frac{X_{[\ldots i]}}{X_{[\ldots i+1]}}$ and set $X' = e_i(-a)X$. Then $X = e_i(a)X'$ is a greedy factorization.*

Proof. Assume $a' > a$. Then there exists k such that $\frac{\Delta_{I_k,J_k}(X)}{\Delta_{I'_k,J_k}(X)} < a'$. If we denote $Y = e_i(-a')X$ then $\Delta_{I_k,J_k}(Y) = \Delta_{I_k,J_k}(X) - a'\Delta_{I'_k,J_k}(X) < 0$, and thus Y cannot be totally nonnegative.

On the other hand, we argue that $X' = e_i(-a)X \in U_{\geq 0}$. By [LPI, Lemma 2.3], it suffices to check nonnegativity of only the row-solid minors of X'. Furthermore it suffices to look at minors with bottom row i, since other row-solid minors do not change when X is multiplied by $e_i(-a)$. But we have $\Delta_{I,J}(X') = \Delta_{I,J}(X) - a\Delta_{I',J}(X)$, where I is a solid minor ending in row i and $I' = (I \setminus \{i\}) \cup \{i+1\}$. By the definition of a and the last statement of Lemma 9.2, we conclude that any such minor in X' is nonnegative. □

One can use Proposition 9.3 to compute the coefficients in the greedy factorization for any finite sequence of Chevalley generators. We illustrate it by the following lemma.

Lemma 9.4. *Let $X \in U_{>0}$ be totally positive.*

(1) *If $X = e_i(a_1)e_{i+1}(a_2)e_i(a_3)X'''$ is a greedy factorization then*

$$a_1 = \frac{X_{[\ldots i-1,i]}}{X_{[\ldots i-1,i+1]}} \qquad a_2 = \frac{X_{[\ldots i-1,i+1]}}{X_{[\ldots i-1,i+2]}} \qquad a_3 = \frac{X_{[\ldots i,i+1]}}{X_{[\ldots i+1,i+2]}} \Big/ \frac{X_{[\ldots i-1,i+1]}}{X_{[\ldots i-1,i+2]}};$$

(2) *if $X = e_{i+1}(a_1)e_i(a_2)e_{i+1}(a_3)X'''$ is a greedy factorization then*

$$a_1 = \frac{X_{[\ldots i,i+1]}}{X_{[\ldots i,i+2]}} \qquad a_2 = \frac{X_{[\ldots i-1,i,i+2]}}{X_{[\ldots i-1,i+1,i+2]}} \qquad a_3 = \frac{X_{[\ldots i-1,i]}}{X_{[\ldots i-1,i+2]}} \Big/ \frac{X_{[\ldots i-1,i,i+2]}}{X_{[\ldots i-1,i+1,i+2]}}.$$

Proof. In the following, we shall write $X^k_{[\ldots I]}$ to mean $\Delta_{I_k,J_k}(X)$, where we assume that some sequence J_k has been fixed, satisfying $J_{k-1} < J_k$ and $|J_k| = k+3$ for each k. When we write $\frac{X^k_{[I]}}{X^k_{[I']}}$ with $|I| = |I'| < 3$ we assume that the initial part of J_k of size $k + |I|$ is used as the column sequence.

We prove the formulae for a_1 and a_2 in the first case first. We already know that if $X = e_i(a_1)X'$ is a greedy factorization then $a_1 = \frac{X_{[\ldots i]}}{X_{[\ldots i+1]}}$. Assume $X' = e_{i+1}(a_2)X''$ is greedy. Then we have

$$\begin{aligned}
a_2 = \frac{X'_{[\ldots i+1]}}{X'_{[\ldots i+2]}} &= \lim_{k\to\infty} \frac{X^k_{[\ldots i-1,i,i+1]}}{X^k_{[\ldots i-1,i,i+2]} - a_1 X^k_{[\ldots i-1,i+1,i+2]}} \\
&= \lim_{k\to\infty} \frac{X^k_{[\ldots i-1,i,i+1]}}{X^k_{[\ldots i-1,i,i+2]} - \frac{X^k_{[\ldots i-1,i]}}{X^k_{[\ldots i-1,i+1]}} X^k_{[\ldots i-1,i+1,i+2]}} \\
&= \lim_{k\to\infty} \frac{X^k_{[\ldots i-1,i,i+1]} X^k_{[\ldots i-1,i+1]}}{X^k_{[\ldots i-1,i,i+2]} X^k_{[\ldots i-1,i+1]} - X^k_{[\ldots i-1,i]} X^k_{[\ldots i-1,i+1,i+2]}} \\
&= \lim_{k\to\infty} \frac{X^k_{[\ldots i-1,i,i+1]} X^k_{[\ldots i-1,i+1]}}{X^k_{[\ldots i-1,i+2]} X^k_{[\ldots i-1,i,i+1]}} = \frac{X_{[\ldots i-1,i+1]}}{X_{[\ldots i-1,i+2]}}.
\end{aligned}$$

The three-term Plücker relation (Lemma 7.1) is used here.

The proof of the formulae for a_1 and a_2 in the second case is similar. Assume now again that $X = e_i(a_1)X'$ is greedy. When we factor $e_{i+1}(a_2)e_i(a_3)$ from X' greedily we get

$$\begin{aligned}
a_3 = \frac{X'_{[\ldots i-1,i,i+2]}}{X'_{[\ldots i-1,i+1,i+2]}} &= \lim_{k\to\infty} \frac{X^k_{[\ldots i-1,i,i+2]} - \frac{X^k_{[\ldots i-1,i]}}{X^k_{[\ldots i-1,i+1]}} X^k_{[\ldots i-1,i+1,i+2]}}{X^k_{[\ldots i-1,i+1,i+2]}} \\
&= \lim_{k\to\infty} \frac{X^k_{[\ldots i-1,i,i+2]} X^k_{[\ldots i-1,i+1]} - X^k_{[\ldots i-1,i]} X^k_{[\ldots i-1,i+1,i+2]}}{X^k_{[\ldots i-1,i+1,i+2]} X^k_{[\ldots i-1,i+1]}} = \frac{X_{[\ldots i-1,i,i+1]} X_{[\ldots i-1,i+2]}}{X_{[\ldots i-1,i+1,i+2]} X_{[\ldots i-1,i+1]}}.
\end{aligned}$$

The proof of the formula for a_3 in the second case is similar. $\square$

9.2. Complete greedy factorizations.

Lemma 9.5. *Let $X \in \Omega$. If $X = e_{\mathbf{i}}(\mathbf{a})$ is greedy, then $\mathbf{i}$ is necessarily an infinite reduced word.*

Proof. Assume $\mathbf{i}$ is not reduced. Take the first initial part ws_i of $\mathbf{i}$ which is not reduced. By the strong exchange condition [Hum, Theorem 5.8] one can find s_j inside w so that $ws_i = us_jvs_i = uv$. Then $s_jv = vs_i$ and using the corresponding braid moves in the factorization of X one can rewrite $X = \ldots e_j(a)e_j(a')\ldots$ where $a' > 0$. This means that the original factor $e_j(a)$ was not greedy – a contradiction implying the lemma. $\square$

Theorem 9.6. *Let $\mathbf{i} \to \mathbf{j}$ be a braid limit of infinite reduced words, and $\mathbf{a} \in \ell^1_{>0}$. Then $X = e_{\mathbf{i}}(\mathbf{a})$ is greedy if and only if $e_{\mathbf{j}}(R^{\mathbf{j}}_{\mathbf{i}}(\mathbf{a}))$ is greedy.*

Proof. By Lemma 5.1, X is totally positive. Greediness is a local property, and thus it suffices to check that it is preserved under braid and commutation relations. In case of commuting s_i and s_j it is clear from Proposition 9.3 that factoring out e_i does not effect the parameter of greedy factorization of e_j and vice versa. For braid moves, by Lemma

9.4, it suffices to check that

$$e_i\left(\frac{X_{[\ldots i]}}{X_{[\ldots i+1]}}\right)e_{i+1}\left(\frac{X_{[\ldots i-1,i+1]}}{X_{[\ldots i-1,i+2]}}\right)e_i\left(\frac{X_{[\ldots i,i+1]}}{X_{[\ldots i+1,i+2]}}/\frac{X_{[\ldots i-1,i+1]}}{X_{[\ldots i-1,i+2]}}\right)$$
$$=e_{i+1}\left(\frac{X_{[\ldots i+1]}}{X_{[\ldots i+2]}}\right)e_i\left(\frac{X_{[\ldots i,i+2]}}{X_{[\ldots i+1,i+2]}}\right)e_{i+1}\left(\frac{X_{[\ldots i]}}{X_{[\ldots i+2]}}/\frac{X_{[\ldots i,i+2]}}{X_{[\ldots i+1,i+2]}}\right).$$

This is straightforward, using the three-term Plücker relations (Lemma 7.1) and (3). □

9.3. **Proof of Lemma 9.2.** Since $X \in U_{>0}$, all the minor ratios in the limit are well-defined.

Roughly speaking, as we let $k \to \infty$ the set J_k grows and moves to the right. We argue that each of the two processes - increasing in size without moving and moving to the right without change in size - does not increase the ratio $\frac{\Delta_{I,J}(X)}{\Delta_{I',J}(X)}$. In fact, it was already shown in [LPI, Lemma 10.5] that if $J \le J'$ have the same cardinality, then

$$\frac{\Delta_{I,J}(X)}{\Delta_{I',J}(X)} \ge \frac{\Delta_{I,J'}(X)}{\Delta_{I',J'}(X)}.$$

To establish Lemma 9.2, it thus remains to consider the case of J increasing in size without moving.

Lemma 9.7. *Suppose that $J' = J \cup J''$ for some set of columns J'' each element of which is bigger than the elements of J. Then*

$$\frac{\Delta_{I,J}(X)}{\Delta_{I',J}(X)} \ge \frac{\Delta_{I_k,J'}(X)}{\Delta_{I'_k,J'}(X)}$$

where $k = |J''|$.

The proof is similar to the one of [LPI, Lemma 10.5] and uses Rhoades and Skandera's Temperley-Lieb immanants (or TL-immanants). These are functions $\mathrm{Imm}_\tau^{\mathrm{TL}}(Y)$ of a $n \times n$ matrix Y, where τ is a Temperley-Lieb diagram. We use the same notation as in [LPI, Section 10.2], referring the reader there, or to [RS] for the definitions.

Theorem 9.8. [RS, Proposition 2.3, Proposition 4.4] *If Y is a totally nonnegative matrix, then* $\mathrm{Imm}_\tau^{\mathrm{TL}}(Y) \ge 0$. *For two subsets $I, J \subset [n]$ of the same cardinality and $S = J \cup (\bar{I})^\wedge$, we have*

$$\Delta_{I,J}(Y) \cdot \Delta_{\bar{I},\bar{J}}(Y) = \sum_{\tau \in \Theta(S)} \mathrm{Imm}_\tau^{\mathrm{TL}}(Y).$$

We also need the following property of Temperley-Lieb immanants.

Lemma 9.9. *If columns i and $i+1$ of a matrix X are equal and the vertices i and $i+1$ in the TL-diagram τ are not matched with each other, then* $\mathrm{Imm}_\tau^{\mathrm{TL}}(X) = 0$.

Proof. Follows from [RS2, Corollary 15] since vertices i and $i+1$ on the column side of a TL-diagram are matched if and only if s_i is a right descent of the corresponding 321-avoiding permutation w. Alternatively, the claim follows immediately from the network interpretation of Temperley-Lieb immanants given in [RS]. □

Proof of Lemma 9.7. Clearly it is enough to prove the lemma for $|J''| = 1$, since the argument can be iterated. Let Y be the submatrix of X induced by the rows in $I \cup I'_1 = I_1 \cup I'$ and columns $J \cup J'$, where we repeat a row or a column if it belongs to both of the sets (that is, $I \cup I'_1$ and $J \cup J'$ are considered as multisets). We index rows and columns of Y again by $I \cup I'_1$ and $J \cup J'$. Whenever there is a repeated column we consider the right one of the two to be in J'. Similarly whenever there is a repeated row we consider the bottom one of the two to be in I'_1. Then $I'_1 = \bar{I}$, $J' = \bar{J}$ and we can apply Theorem 9.8 to the products $\Delta_{I,J}(X)\Delta_{I'_1,J'}(X)$ and $\Delta_{I',J}(X)\Delta_{I_1,J'}(X)$.

By Lemma 9.9, all Temperley-Lieb immanants of Y in which vertices of J are not matched with their counterparts in J' are zero. Thus we can restrict our attention to the immanants whose diagrams have this property. The rest of the points consist of $I \cup I'_1$ and the single point from J'', which we think of as lying on a line arranged from left to right. The points in $I \cup I'$ are colored so that in any initial subsequence (reading from the left) the number of white points is at least as large as the number of black points. The colorings corresponding to $\Delta_{I,J}(X)\Delta_{I'_1,J'}(X)$ and $\Delta_{I',J}(X)\Delta_{I_1,J'}(X)$ agree on all vertices but two: the rightmost one, coming from J'', and the leftmost one, added when passing from I' to I'_1 (or from I to I_1). The first product corresponds to coloring leftmost vertex black and rightmost vertex white, while the second product has colors the other way around. Now it is easy to see that any non-crossing matching compatible with the second product is also compatible with the first one, since in the former the leftmost vertex has to be matched with the rightmost vertex. The claim of the lemma now follows from Theorem 9.8. $\square$

Proof of Lemma 9.2. It follows from [LPI, Lemma 10.5] and Lemma 9.7 that

$$\frac{\Delta_{I_k,J_k}(X)}{\Delta_{I'_k,J_k}(X)} \geq \frac{\Delta_{I_{k+1},J_{k+1}}(X)}{\Delta_{I'_{k+1},J_{k+1}}(X)}$$

and thus the limit exists and is not bigger than each individual fraction $\frac{\Delta_{I_k,J_k}(X)}{\Delta_{I'_k,J_k}(X)}$. The argument for the independence on choice of $\{J_k\}$ is similar to that in [LPI, Theorem 10.6]. Namely, for two different choices of column sequence and for a particular term in one of them, one can always find a smaller term in the other one. This implies the limits cannot be different. $\square$

10. Open problems and conjectures

From Section 3.

In [GLS], Geiss, Leclerc, and Schröer studied (in the Kac-Moody setting) the cluster algebra structure of the coordinate ring $\mathbb{C}[U^w]$ of the unipotent cells U^w obtained by intersecting U^{pol} with the Bruhat cells. Their work should be compared with our Proposition 3.3, which gives a complete list of positive minors for totally nonnegative elements in each unipotent cell. Presumably, the cluster variables in each cluster for $\mathbb{C}[U^w]$ give rise to a minimal set of totally positive criteria (cf. [FZ]).

From Section 4.

Problem 10.1. Characterize, in terms of infinite reduced words, the minimal elements of the limit weak order for other affine types.

Question 10.2. Are minimal elements of limit weak order necessarily fully commutative?

In other affine types infinite Coxeter elements are still reduced [KP], but may not necessarily be fully commutative. Note also that in affine type B there are *more* minimal blocks than Coxeter elements.

From Section 5.

We conjecture that the inclusion relations of the $E_{\mathbf{i}}$ is exactly the limit weak order.

Conjecture 10.3. *The converse of the Corollary 5.7 holds: if $E_{[\mathbf{j}]} \subset E_{[\mathbf{i}]}$ then $[\mathbf{i}] \leq [\mathbf{j}]$. Furthermore, if $[\mathbf{i}] < [\mathbf{j}]$ then the containment $E_{[\mathbf{j}]} \subset E_{[\mathbf{i}]}$ is strict.*

Problem 10.4. Describe completely the pairs $([\mathbf{i}], [\mathbf{j}])$ such that $E_{[\mathbf{i}]}$ and $E_{[\mathbf{j}]}$ have non-trivial intersection. Describe these intersections completely.

Problem 10.4 might be solved by an affirmative answer to the following question.

Question 10.5. Assume $e_{\mathbf{i}}(\mathbf{a}) = X = e_{\mathbf{j}}(\mathbf{b})$. Can the equality $e_{\mathbf{i}}(\mathbf{a}) = e_{\mathbf{j}}(\mathbf{b})$ always be proved using braid limits? What about braid limits where one is allowed to go through intermediate non-reduced products, that is one is allowed to "split" the Chevalley generators as in Example 6.1?

An affirmative solution to Question 10.5 may involve a long sequence of braid limits: if $e_{\mathbf{i}}(\mathbf{a}) = X = e_{\mathbf{j}}(\mathbf{b})$ there does not always exist a factorization $X = e_{\mathbf{k}}(\mathbf{c})$ and braid limits $\mathbf{k} \to \mathbf{i}$ and $\mathbf{k} \to \mathbf{j}$, such that $\mathbf{a} = R^{\mathbf{i}}_{\mathbf{k}}(\mathbf{c})$ and $\mathbf{b} = R^{\mathbf{j}}_{\mathbf{k}}(\mathbf{c})$. For example, by Proposition 5.3 this is the case if $\mathbf{i} = \mathbf{j}$ but $\mathbf{a} \neq \mathbf{b}$. Also there does not always exist a factorization $X = e_{\mathbf{k}}(\mathbf{c})$ and braid limits $\mathbf{i} \to \mathbf{k}$ and $\mathbf{j} \to \mathbf{k}$, such that $\mathbf{c} = R^{\mathbf{k}}_{\mathbf{i}}(\mathbf{a})$ and $\mathbf{b} = R^{\mathbf{k}}_{\mathbf{j}}(\mathbf{b})$. For example consider the situation in Example 5.3.

It would also be interesting to describe the topology of each $E_{\mathbf{i}}$, and of their intersections (cf. Theorem 1.1).

From Section 6. Let $X \in \Omega$ and let $I(X)$ be the set of equivalence classes $[\mathbf{i}]$ of infinite reduced words $\mathbf{i}$ such that $X \in E_{\mathbf{i}}$. It is clear from Corollary 5.7 and Theorem 5.5 that $I(X)$ is a lower order ideal in limit weak order. Question 10.5 partly motivates (but is not implied by) the following conjecture.

Conjecture 10.6 (Principal ideal conjecture)**.** *For any $X \in \Omega$, the ideal $I(X)$ is a principal ideal.*

One special case of Conjecture 10.6 is $X \in A_{c^\infty}$. In this case, it may be reasonable to conjecture that $I(X) = \{[c^\infty]\}$; that is, these matrices only have a single factorization, which is an infinite Coxeter factorization (cf. Proposition 6.9). Similarly, it may be reasonable to conjecture that if $X \in B_{c^\infty}$ then X has a factorization other than the c^∞ factorization. This is consistent with Example 6.1.

We briefly explain some consequences and variations of the Principal ideal conjecture. The following conjecture is a combinatorial consequence of Conjecture 10.6.

Conjecture 10.7. *Suppose $X \in E_{\mathbf{i}}$ and $X \in E_{\mathbf{j}}$. Then the join $[\mathbf{i}] \vee [\mathbf{j}]$ exists in the limit weak order.*

The condition that the join $[\mathbf{i}] \vee [\mathbf{j}]$ exists can be made more precise.

Proposition 10.8. *Let $\mathbf{i}$ and $\mathbf{j}$ be two infinite reduced words. Then the join $[\mathbf{i}] \vee [\mathbf{j}]$ exists in the limit weak order if and only if there does not exist a (finite) root $\alpha \in \Delta_0^+$ such that both α and $\delta - \alpha$ lie in* $\mathrm{Inv}(\mathbf{i}) \cup \mathrm{Inv}(\mathbf{j})$.

Proof. Given $I = \mathrm{Inv}(\mathbf{i}) \cup \mathrm{Inv}(\mathbf{j})$, we define a partial order $\preceq$ on $[n]$ as follows. For each $1 \le a < b \le n$, set $a \prec b$ whenever $\alpha_{ab} \in I$, and $b \prec a$ whenever $\delta - \alpha_{ab} \in I$. Transitivity and the fact that this is a partial order (rather than a preorder) follows from Proposition 4.1, and the assumption that α_{ab} and $\delta - \alpha_{ab}$ are not simultaneously in I. Now pick any total order $\preceq'$ which extends the partial order $\preceq$, consider $\preceq'$ as defining a maximal face of the braid arrangement, and let $[\mathbf{k}] \in \tilde{\mathcal{W}}$ be the corresponding element under the bijection of Theorem 4.11. Note that $[\mathbf{k}]$ is a maximal element of $\tilde{\mathcal{W}}$, and that it is an upper bound for $[\mathbf{i}]$ and $[\mathbf{j}]$. □

Recall that in Conjecture 6.3 we conjectured that if $\mathbf{i}$ is an infinite reduced word which is minimal in its block then $e_{\mathbf{i}}$ is injective.

Proposition 10.9. *Conjecture 10.7 implies Conjecture 6.3.*

Proof. We know that an element $[\mathbf{i}]$ minimal in its block has a representative $\mathbf{i} = t_\lambda^\infty$ for some translation element t_λ. Choose a reduced expression $t_\lambda = s_{j_1} \ldots s_{j_l}$. Assume $e_{\mathbf{i}}$ is not injective, that is, we have $X = e_{\mathbf{i}}(\mathbf{a}) = e_{\mathbf{i}}(\mathbf{a}')$ for $\mathbf{a} \neq \mathbf{a}'$. We may assume that $a_1 \neq a_1'$. For otherwise, we may write $X = e_w(a_1, a_2, \ldots, a_r) e_{\mathbf{i}'}(a_{r+1}, \ldots) = e_w(a_1, a_2, \ldots, a_r) e_{\mathbf{i}'}(a'_{r+1}, \ldots)$. The infinite reduced word $\mathbf{i}'$ is equal to $(t_{v^{-1}\cdot\lambda})^\infty$, where $w = vt_\mu$ for some $v \in W$ and $\mu \in Q^\vee$. In particular, $[\mathbf{i}']$ is minimal in its block.

Now, without loss of generality assume $a_1 < a_1'$. Then $e_{j_1}(-a_1)X$ belongs to both $E_{t_\lambda^\infty}$ and $E_{t^\infty_{s_{j_1}\cdot\lambda}}$. Since α_{j_1} is an inversion of t_λ, we have $\langle \alpha_{j_1}, \lambda \rangle < 0$ (see the proof of Proposition 4.3). But then $\langle \alpha_{j_1}, s_{j_1}\lambda \rangle > 0$, so $\delta - \alpha_{j_1}$ is an inversion of $t_{s_{j_1}\cdot\lambda}$. Assuming Conjecture 10.7, this contradicts Proposition 10.8. □

A problem significantly harder than Conjecture 6.3 is

Problem 10.10. For each $X \in \Omega$ and infinite reduced word $\mathbf{i}$, completely describe $e_{\mathbf{i}}^{-1}(X)$.

From Section 7.

Question 10.11. Assume $X \in \Omega$ lies in the ASW cell $A(w, v)$. For a fixed choice of i, in which ASW cells may the matrix $e_i(a)X$ lie as a assumes all positive values? More generally, where may YX lie if $Y \in E_u$ for a fixed $u \in \tilde{W}$?

Let $X \in \Omega$. ASW factorization gives rise to a distinguished factorization of X. But whirl ASW factorization (Remark 7.5) gives rise to another distinguished factorization. Can we get every factorization of X using a mixture of these operations?

Question 10.12. Let $X \in \Omega$. Apply to X ASW or whirl ASW factorization repeatedly, choosing freely which of the two to apply at each step. Is it true that for every $[\mathbf{i}] \in I(X)$ one can find a sequence of ASW or whirl ASW choices that shows $X \in E_{[\mathbf{i}]}$?

In Proposition 5.2, we showed that every $X = e_{\mathbf{i}}(\mathbf{a})$ for $\mathbf{i}$ not necessarily reduced lies in $\Omega \cup U_{\ge 0}^{\mathrm{pol}}$. One can obtain a (possibly finite) reduced word $\mathbf{j}$ from $\mathbf{i}$, as follows. Recall that the Demazure product is defined by

$$w \circ s_i = \begin{cases} ws_i & ws_i > w, \\ w & \text{otherwise.} \end{cases}$$

The Demazure product is associative. We define the reduction $\mathbf{j} = j_1 j_2 \cdots$ of $\mathbf{i} = i_1 i_2 \cdots$ by requiring that $\mathbf{j}$ is reduced and that the list $s_{j_1}, s_{j_1}s_{j_2}, s_{j_1}s_{j_2}s_{j_3}, \ldots$, coincides with $s_{i_1}, s_{i_1} \circ s_{i_2}, s_{i_1} \circ s_{i_2} \circ s_{i_3}, \ldots$, after repetitions are removed.

Question 10.13. Assume that $\mathbf{i}$ is an infinite non-reduced word and $X \in E_{\mathbf{i}}$. Let $\mathbf{j}$ be the reduction of $\mathbf{i}$. Is it true that $X \in E_{\mathbf{j}}$?

Given a factorization $X = e_{\mathbf{i}}(\mathbf{a})$, one can attempt to produce a factorization $X = e_{\mathbf{j}}(\mathbf{a}')$ by "adding" each of the generators $e_{i_r}(a_r)$ one at a time. However, when the product is not reduced, many previously calculated parameters may change when the additional factor $e_{i_r}(a_r)$ is introduced. A priori, we have no guarantee that in the limit some of the parameters do not go to 0.

Example 10.1. The simplest example is a non-reduced product that starts

$$X = e_1(a)e_2(b)e_1(c)e_2(d)\ldots.$$

Then when we multiply by the fourth factor, it gets absorbed into the previous three factors as follows: $e_1(a)e_2(b)e_1(c)e_2(d) = e_1(a+\frac{cd}{b+d})e_2(b+d)e_1(\frac{bc}{b+d})$. The third parameter has decreased from c to $\frac{bc}{b+d}$.

From Section 8.

Infinite products of Chevalley generators also make sense for general Kac-Moody groups. We intend to study them in the future [LPKM].

Conjecture 10.14. *The TP Exchange Lemma (Theorem 8.1) holds in Kac-Moody generality.*

From Section 9.

For a finite reduced word $\mathbf{i}$, Berenstein and Zelevinsky [BZ] gave an expression for the parameters $\mathbf{a}$ in the matrix $X = e_{\mathbf{i}}(\mathbf{a})$, in terms of the minors of (the twist matrix of) X. Because of the lack of injectivity of $e_{\mathbf{i}}$ in the case that $\mathbf{i}$ is infinite, this problem cannot be easily posed in our setting. However, it does make sense if we restrict to greedy factorizations.

Problem 10.15. Let $\mathbf{i}$ be an infinite reduced word. Assume that $X \in \Omega$ has a greedy factorization $e_{\mathbf{i}}(\mathbf{a})$. Find an explicit formula for the parameters a_j in the spirit of Lemma 9.4. Find an explicit formula that is manifestly positive.

Example 10.2. Let $n = 3$ and suppose that a greedy factorization of X starts with $e_1(a)e_2(b)e_0(c)\ldots$. Then it can be computed that

$$c = \frac{X_{[\ldots -1,0,3]}}{X_{[\ldots -1,0,5]}} \Big/ \left(\frac{X_{[\ldots -1,0,4]}}{X_{[\ldots, -1,0,5]}} - \frac{X_{[\ldots -1,0,1]}}{X_{[\ldots -1,0,2]}}\right).$$

While explicit, this expression is not manifestly positive. One can use the Temperley-Lieb immanants of Section 9.3 to prove the positivity of the denominator. However, it seems desirable to have an expression which is manifestly positive in terms of the minors of X.

Let $J(X)$ be the set of equivalence classes $[\mathbf{i}]$ of infinite reduced words $\mathbf{i}$ such that X has a greedy factorization of the form $e_{\mathbf{i}}(\mathbf{a})$.

Question 10.16. Is it true that $I(X) = J(X)$ for any $X \in \Omega$? Equivalently, if X has a factorization of the form $e_{\mathbf{i}}(\mathbf{a})$, does it necessarily have a greedy factorization of the same form?

By Theorem 9.6 and Theorem 5.5, $J(X)$ is an ideal in limit weak order.

Conjecture 10.17. *The ideal $J(X)$ is principal.*

One can also state an analog of the weaker Conjecture 10.7 for greedy factorizations.

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